ENGINEERING TRIPOS PART IIA

Thursday 1 May 2008 9 to 10.30

Module 3C7

MECHANICS OF SOLIDS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments:
3C7 datasheet (2 pages)

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Single-sided script paper	Engineering Data Book
Graph paper	CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) Define the term 'plane stress' in relation to two-dimensional problems in the mechanics of solids. [15%]

(b) A thin unconstrained disc of material of external radius b contains a central circular hole of radius a. The temperature of the disc is increased from T_0 to T. If the linear coefficient of thermal expansion of the disc is α, determine the resulting displacement at the edge of the hole relative to the centre of the hole. [15%]

(c) The front brake disc of a railway carriage can be modelled as a solid circular disc of radius 250 mm. During braking an outer annular ring of the disk of width 100 mm is heated uniformly to a temperature of 150 °C, while the remainder of the disc stays at a temperature of 25 °C. Assuming plane stress conditions, determine the stresses at the centre of the disc. The disc is made of a cast iron with the following properties: Young's Modulus $E = 180$ GPa, Poisson's ratio $\nu = 0.28$, $\alpha = 12\times10^{-6}\,\mathrm{K}^{-1}$. [70%]

2 A long circular cylinder has an inner radius a and outer radius b . It is made of material with a shear yield stress k.

(a) Show that the pressure difference Δp between the inner and outer surfaces which will just cause the inner surface of the vessel to yield, assuming plane-stress conditions and the Tresca yield criterion, is given by the expression

$$\frac{\Delta p}{2k} = \frac{m^2 - 1}{2m^2} \quad \text{where} \quad m = b/a .$$ [30%]

(b) Obtain the corresponding expression for the pressure difference which will just bring the plastic zone to the outer surface. [30%]

(c) Two long cylinders are to be shrunk together to form a pressure vessel. The inner and outer diameters of the inner cylinder are 200 mm and 300 mm respectively. Find the outer diameter D of the outer cylinder so that, under an internal pressure of 240 MPa, the inner cylinder yields up to a diameter of 270 mm and the outer cylinder is at the point of yield at its inner diameter. Assume a Tresca yield criterion with a shear yield stress of 210 MPa for both cylinders. [40%]

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3 The expression

$$\phi = \frac{\tau_0}{4}\left\{xy - \frac{xy^2}{h} - \frac{xy^3}{h^2} + \frac{Ly^2}{h} + \frac{Ly^3}{h^2}\right\}$$

which satisfies the biharmonic equation is to be used as an Airy stress function to describe the state of stress within the cantilever shown in Fig. 1, which is loaded by a uniform shear stress of magnitude τ_0 along its upper surface.

(a) Obtain expressions for the stress components at some general point (x,y). [25%]

(b) Verify that these stress components are consistent with the specified boundary conditions on the upper and lower surfaces. [25%]

(c) Sketch curves showing how the direct and shear stresses derived from this function are distributed across the root of the cantilever, and investigate the extent to which they are consistent with overall equilibrium. [25%]

(d) By drawing a Mohr's circle, or otherwise, calculate the maximum principal tensile stress at ($x = 0$, $y = h$) and compare the result with the predictions of simple beam theory. [25%]

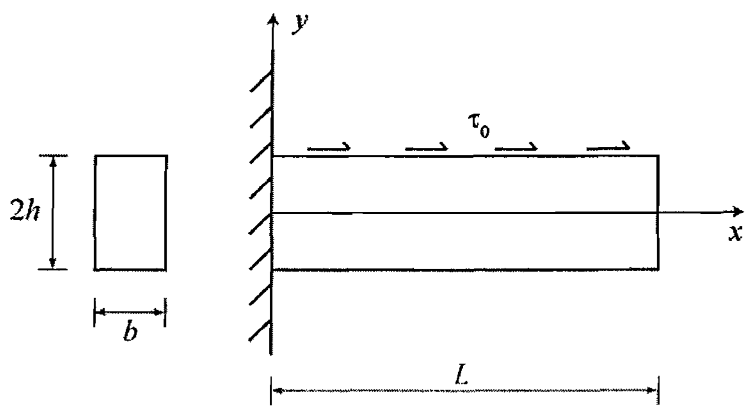

Fig. 1

4 (a) State the upper and lower bound theorems of plasticity and comment briefly on their practical applicability. [30%]

(b) Figure 2 shows a section of a thick-walled cylindrical pressure vessel whose inner radius is a and outer radius b. The vessel contains gas at a gauge pressure p. A failure mechanism is to be considered which consists of a set of intersecting shear planes as indicated. Each pair of planes is symmetrical and meets the inner surface of the vessel tangentially; the planes make angles of 2β where they intersect. The material of the vessel can be considered to be rigid-perfectly plastic with a flow stress in shear of magnitude k.

(i) Show that the distance PQ is given by

$$\mathrm{PQ}^2 = (b-a)(b+a)$$ [10%]

(ii) For the special case when $2\beta = 90°$, estimate the failure pressure p_{max} in terms of k and the ratio b/a. [25%]

(iii) For the general case when the angle at which the shear planes intersect can vary, investigate the influence of the angle β on the failure pressure p_{max}. [35%]

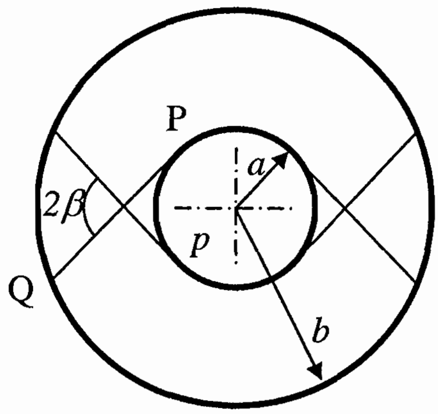

Fig. 2

END OF PAPER

Module 3C7: Mechanics of Solids

ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

Discs and tubes

Equilibrium

$$\sigma_{\theta\theta} = \frac{d(r\sigma_{rr})}{dr} + \rho\omega^2 r^2$$

Lamé's equations (in elasticity)

$$\sigma_{rr} = A - \frac{B}{r^2} - \frac{3+\nu}{8}\rho\omega^2 r^2 - \frac{E\alpha}{r^2}\int_c^r rT\,dr$$

$$\sigma_{\theta\theta} = A + \frac{B}{r^2} - \frac{1+3\nu}{8}\rho\omega^2 r^2 + \frac{E\alpha}{r^2}\int_c^r rT\,dr - E\alpha T$$

Spheres

Equilibrium

$$\sigma_{\theta\theta} = \frac{1}{2r}\frac{d(r^2\sigma_{rr})}{dr}$$

Lamé's equations (in elasticity)

$$\sigma_{rr} = A - \frac{B}{r^3}$$

$$\sigma_{\theta\theta} = A + \frac{B}{2r^3}$$

2. Plane stress and plane strain

Cartesian coordinates

Strains

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Compatibility

$$\frac{\partial^2\gamma_{xy}}{\partial x\partial y} = \frac{\partial^2\varepsilon_{xx}}{\partial y^2} + \frac{\partial^2\varepsilon_{yy}}{\partial x^2}$$

or (in elasticity)

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right\}(\sigma_{xx} + \sigma_{yy}) = 0$$

Equilibrium

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} = 0$$

$$\frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{xy}}{\partial x} = 0$$

$^4\phi = 0$ (in elasticity)

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right\}\left\{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right\} = 0$$

Airy Stress Function

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2}$$

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2}$$

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$$

Polar coordinates

Strains

$$\varepsilon_{rr} = \frac{\partial u}{\partial r}$$

$$\varepsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial\theta}$$

$$\gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial\theta} - \frac{v}{r}$$

Compatibility

$$\frac{\partial}{\partial r}\left\{r\frac{\partial\gamma_{r\theta}}{\partial\theta}\right\} = \frac{\partial}{\partial r}\left\{r^2\frac{\partial\varepsilon_{\theta\theta}}{\partial r}\right\} - r\frac{\partial\varepsilon_{rr}}{\partial r} + \frac{\partial^2\varepsilon_{rr}}{\partial\theta^2}$$

or (in elasticity)

$$\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right\}(\sigma_{rr} + \sigma_{\theta\theta}) = 0$$

Equilibrium

$$\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{\partial\sigma_{r\theta}}{\partial\theta} - \sigma_{\theta\theta} = 0$$

$$\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$$

$^4\phi = 0$ (in elasticity)

$$\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right\} \infty \left\{\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right\} = 0$$

Airy Stress Function

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial\phi}{d\theta}\right\}$$

3. Torsion of prismatic bars

Prandtl stress function: $\sigma_{zx}\ (= \tau_x) = \frac{dF}{dy}$, $\sigma_{zy}\ (= \tau_y) = -\frac{dF}{dx}$

Equilibrium: $T = 2\int_A F\,dA$

Governing equation for elastic torsion: $\nabla^2 F = -2G\beta$ where β is the angle of twist per unit length.

4. Total potential energy of a body

$$\Pi = U - W$$

where $U = \frac{1}{2}\int_V \underset{\sim}{\varepsilon}^T [D]\, \underset{\sim}{\varepsilon}\, dV$, $W = \underset{\sim}{P}^{\,T} \underset{\sim}{u}$ and $[D]$ is the elastic stiffness matrix.

5. Principal stresses and stress invariants

Values of the principal stresses, σ_P, can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx} - \sigma_P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma_P & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma_P \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_P.

Expanding: $\sigma_P^3 - I_1\,\sigma_P^2 + I_2\sigma_P - I_3 = 0$ where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}.$$

6. Equivalent stress and strain

Equivalent stress $\bar{\sigma} = \sqrt{\frac{1}{2}}\ \{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\}^{1/2}$

Equivalent strain increment $d\bar{\varepsilon} = \sqrt{\frac{2}{3}}\ \{d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2\}^{1/2}$

7. Yield criteria and flow rules

Tresca

Material yields when maximum value of $|\sigma_1 - \sigma_2|$, $|\sigma_2 - \sigma_3|$ or $|\sigma_3 - \sigma_1| = Y = 2k$, and then,

if σ_3 is the intermediate stress, $d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

von Mises

Material yields when, $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{d\varepsilon_1}{\sigma'_1} = \frac{d\varepsilon_2}{\sigma'_2} = \frac{d\varepsilon_3}{\sigma'_3} = \frac{d\varepsilon_1 - d\varepsilon_2}{\sigma_1 - \sigma_2} = \frac{d\varepsilon_2 - d\varepsilon_3}{\sigma_2 - \sigma_3} = \frac{d\varepsilon_3 - d\varepsilon_1}{\sigma_3 - \sigma_1} = \lambda = \frac{3}{2}\frac{d\bar{\varepsilon}}{\bar{\sigma}}.$$