ENGINEERING TRIPOS PART IIA

Thursday 8 May 2008 2.30 to 4.00

Module 3D7

FINITE ELEMENT METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: Special datasheets (3 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Version 3

1 A planar pin-jointed structure, with geometry as shown in Fig. 1, consists of four straight members. Each member behaves linearly elastically and has axial stiffness EA.

(a) Set up an equilibrium matrix $\mathbf{H}$ which relates the axial forces in the structure $\mathbf{r} = \{t_I \quad t_{II} \quad t_{III} \quad t_{IV}\}^T$ to the set of external loads $\mathbf{p} = \{p_{AX}\, p_{AY}\}^T = \{W \quad 0\}^T$ applied to node A. [20%]

(b) Obtain a general solution to the set of equations $\mathbf{H}\,\mathbf{r} = \mathbf{p}$. [40%]

(c) Evaluate $\mathbf{r}$ if the structure is initially unstressed. [40%]

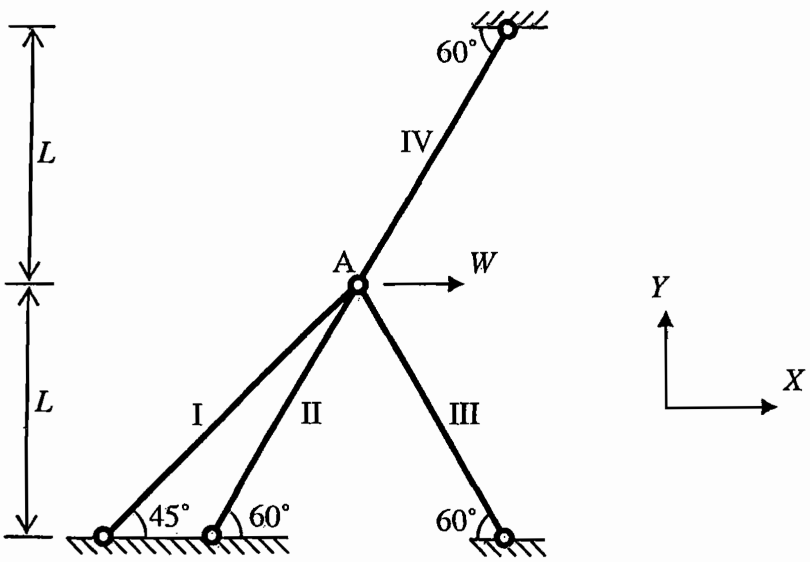


Fig. 1

2 Figure 2 shows a finite element mesh of three-node triangles modelling a thin plate subject to in-plane gravitational loading g in the direction shown. The plate has a thickness of 1 mm and a density of 3000 kg/m^3. You may assume that $g = 10$ m/s^2.

(a) Find the equivalent nodal load components p_{9X} and p_{9Y}. [30%]

(b) Hence determine the equivalent nodal load components at all the other nodes. [30%]

(c) In a subsequent calculation all the three-node triangular elements are replaced with six-node triangular elements by introducing additional mid-side nodes. What qualitative change would there be in the equivalent nodal loads p_{9X} and p_{9Y} at the node numbered 9 in the original mesh? Explain your reasoning but do not perform additional calculations. [20%]

(d) Briefly discuss the advantages and disadvantages of this change of elements. [20%]

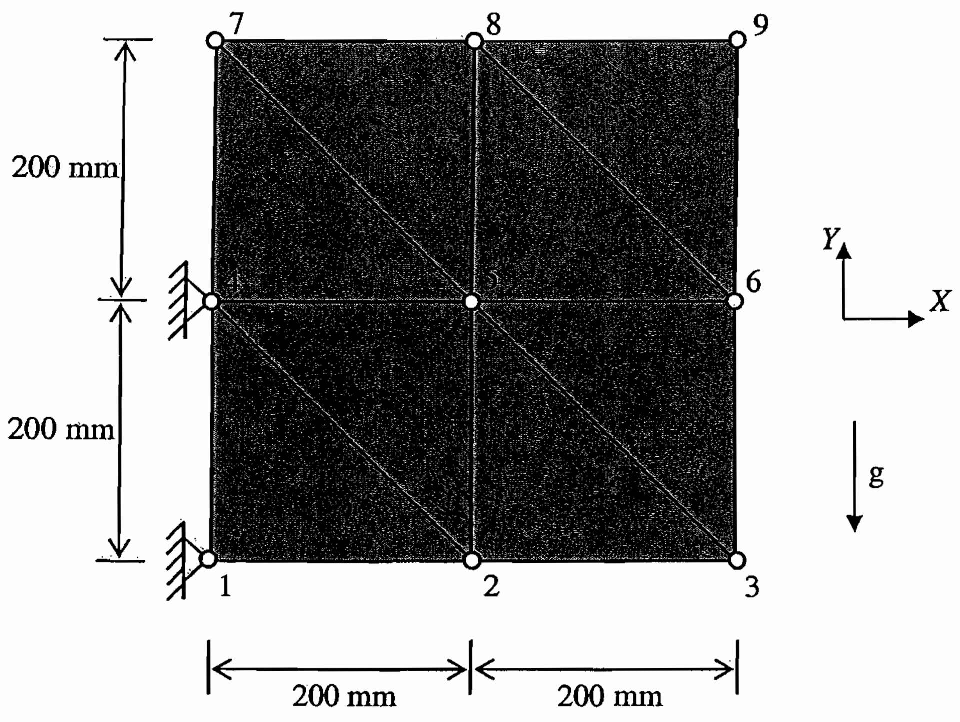


Fig. 2

3 The temperature profile inside a bar shown in Fig. 3(a) is a concern. The bar has three segments each of 1 metre length. Each segment has a different cross-sectional area A and thermal conductivity k as shown in the figure. The temperature at the left hand end is 50 °C and the flux at the right hand end is 15 $\mathrm{Jm^{-2}s^{-1}}$. The side boundaries are insulated.

(a) The problem can be simplified to a one-dimensional steady-state heat transfer problem. The governing equation can be expressed as follows:

$$\frac{d}{dx}\left(Ak\frac{dT}{dx}\right)=0$$

where T is the temperature, A is the cross-sectional area, and k is the thermal conductivity. Show that the weak formulation of the governing equation including the boundary conditions is

$$\int_0^3 \frac{dv}{dx}Ak\frac{dT}{dx}dx = 0.3(v)_{x=0}q_0 - 0.75(v)_{x=3}$$

where v is a weight function and q_0 is the flux at $x=0$ (the left hand end). [20%]

(b) The temperature T and the weight function v are approximated using the following shape functions.

$$T=\mathbf{Na}, \quad \frac{dT}{dx}=\frac{d\mathbf{N}}{dx}\mathbf{a}=\mathbf{Ba}$$

$$v=\mathbf{Nc}, \quad \frac{dv}{dx}=\frac{d\mathbf{N}}{dx}\mathbf{c}=\mathbf{Bc}$$

where $\mathbf{N}$ is the shape function matrix, $\mathbf{a}$ is the vector of nodal temperature values and $\mathbf{c}$ is the vector of arbitrary nodal values. Show that the finite element approximation of the weak form given in part (a) becomes

$$\left(\int_0^3 \mathbf{B}^{\mathrm{T}}Ak\mathbf{B}dx\right)\mathbf{a} = 0.3(\mathbf{N}^{\mathrm{T}})_{x=0}q_0 - 0.75(\mathbf{N}^{\mathrm{T}})_{x=3}$$ [20%]

(c) For the finite element approximation a mesh shown in Fig. 1(b) is used with the following linear shape functions for each element.

Element 1 (Nodes 1 & 2) :	$N_1 = [(1-x), x]$	for $0 \le x \le 1$ and $= 0$ otherwise
Element 2 (Nodes 2 & 3) :	$N_2 = [(2-x), (x-1)]$	for $1 \le x \le 2$ and $= 0$ otherwise
Element 3 (Nodes 3 & 4) :	$N_3 = [(3-x), (x-2)]$	for $2 \le x \le 3$ and $= 0$ otherwise

Derive the stiffness matrix for each element. [20%]

(d) Develop the matrix form of the weak formulation derived in part (b). There is no need to solve for the actual nodal values. [20%]

(e) The one dimensional model cannot provide a solution to local variations of temperature around the joints of the segments. Sketch the shape of the temperature contours that you would expect to see around the joints. [20%]

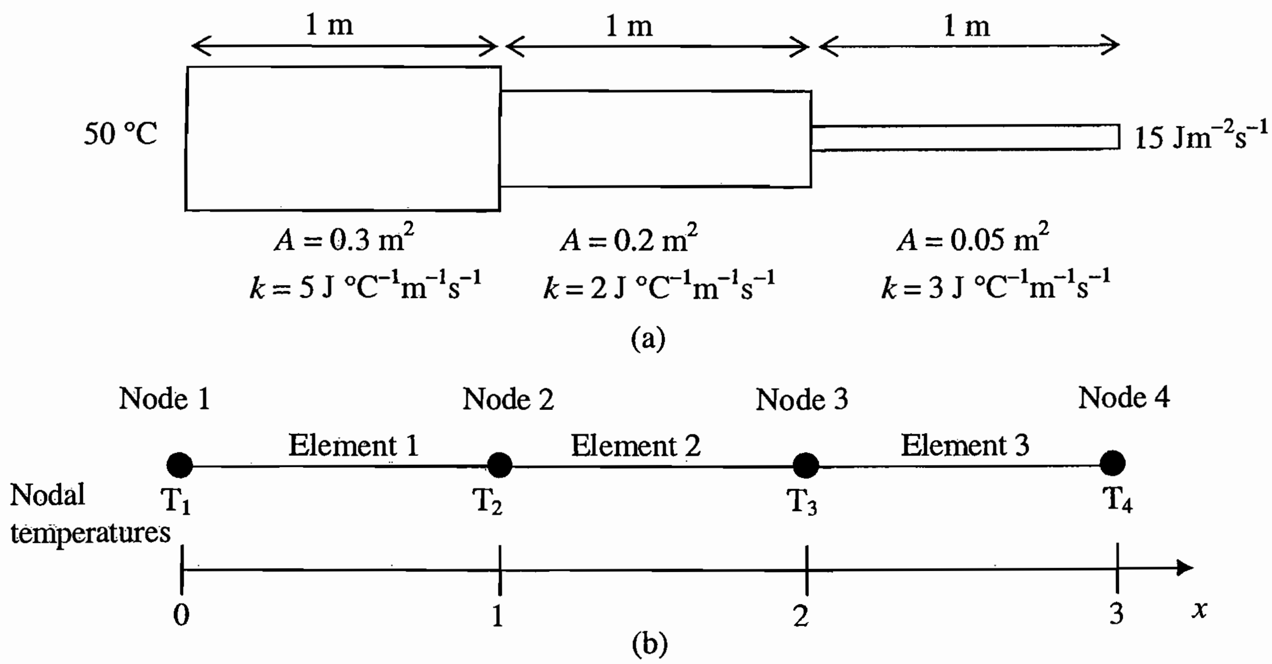


Fig. 3

4 A finite element mesh with boundary conditions as shown in Fig. 4 consists of a single four-node bi-linear quadrilateral element. The element represents part of a planar sheet of material with Young's modulus E = 100 kN/mm^2, Poisson's ratio ν = 0 and thickness t = 1 mm. The origin of the co-ordinate system is at the centre of the element.

(a) Find the shape functions for the element and hence formulate the strain shape function matrix **B**. You need only consider the unconstrained degrees of freedom. [40%]

(b) Given the additional boundary conditions $d_{3X} = d_{3Y} = 0$ calculate the stiffness matrix **K** that relates the nodal displacement $\mathbf{d} = [d_{2X}]$ to the corresponding nodal force $\mathbf{p} = [p_{2X}]$. [60%]

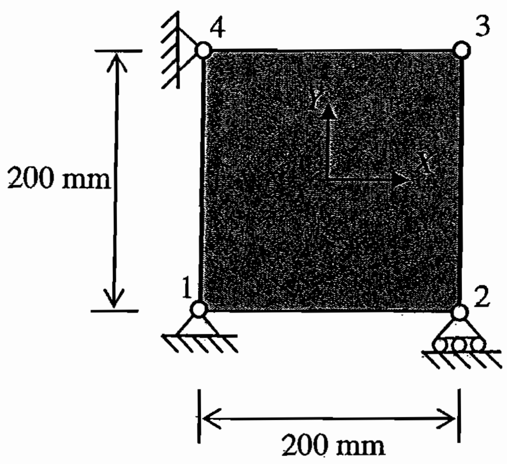


Fig. 4

END OF PAPER

Part IIA: Module 3D7 2007/08

Finite Element Methods

Formulae

Force Method

Stress resultants: solve $\mathbf{Hr} = \mathbf{p}$ and find $\mathbf{r} = \mathbf{r}_0 + \mathbf{Sx}$;
then, solve $\mathbf{S}^T\mathbf{FSx} = -\mathbf{S}^T(\mathbf{Fr}_0 + \mathbf{e}_0)$ for $\mathbf{x}$.

Displacements: solve $\mathbf{H}^T\mathbf{d} = \mathbf{e}$, where $\mathbf{e} = \mathbf{Fr} + \mathbf{e}_0$.

Displacement Method

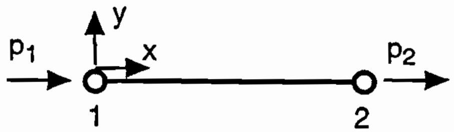
Displacements: solve $\mathbf{Kd} = \mathbf{p}$.

Stress resultants: for element i, solve $\mathbf{F}_i\mathbf{r}_i = \mathbf{e}_i$, where $\mathbf{e}_i = (\mathbf{H}_i')^T\mathbf{d}_i'$.

SP
February 2004

DDS
March 2008

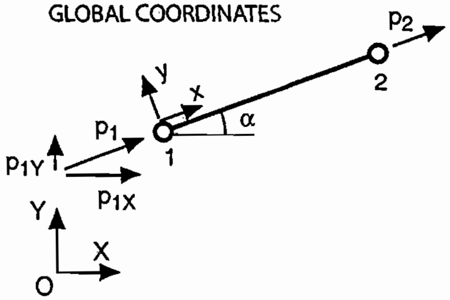
PIN-JOINTED BAR in LOCAL COORDINATES



Static variables	Kinematic variables	Equilibrium	Elasticity	Stiffness $\mathbf{K}_i = \mathbf{H}_i\mathbf{F}_i^{-1}\mathbf{H}_i^T$
$\mathbf{r}_i = [t]$ t = axial force $\mathbf{p}_i = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$	$\mathbf{e}_i = [e]$ e = extension $\mathbf{d}_i = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$	$\mathbf{H}_i = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$	$\mathbf{F}_i = [a]$	$\mathbf{K}_i = \begin{bmatrix} 1/a & -1/a \\ -1/a & 1/a \end{bmatrix}$
Equilibrium Compatibility Constitutive Stiffness	$\mathbf{H}_i\mathbf{r}_i = \mathbf{p}_i$ $\mathbf{H}_i^T\mathbf{d}_i = \mathbf{e}_i$ $\mathbf{F}_i\mathbf{r}_i + \mathbf{e}_{i0} = \mathbf{e}_i$ $\mathbf{K}_i\mathbf{d}_i = \mathbf{p}_i$			

$a = L/AE$, AE = axial stiffness

PIN-JOINTED BAR in GLOBAL COORDINATES



Static variables	Kinematic variables	Coordinate transformation	Equilibrium	Stiffness
$\mathbf{r}_i$ $\mathbf{p}'_i = \begin{bmatrix} p_{1X} \\ p_{1Y} \\ p_{2X} \\ p_{2Y} \end{bmatrix}$	$\mathbf{e}_i$ $\mathbf{d}'_i = \begin{bmatrix} d_{1X} \\ d_{1Y} \\ d_{2X} \\ d_{2Y} \end{bmatrix}$	$\mathbf{T}_i = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}$ $\mathbf{R} = \begin{bmatrix} u \\ v \end{bmatrix}$	$\mathbf{H}'_i = \begin{bmatrix} -u \\ -v \\ u \\ v \end{bmatrix}$	$\mathbf{K}'_i = \frac{1}{a}\begin{bmatrix} u^2 & uv & -u^2 & -uv \\ & v^2 & -uv & -v^2 \\ & & u^2 & uv \\ \text{symm.} & & & v^2 \end{bmatrix}$
Equilibrium Compatibility Constitutive Stiffness Transformations	$\mathbf{H}'_i\mathbf{r}_i = \mathbf{p}'_i$ $\mathbf{H}'^T_i\mathbf{d}'_i = \mathbf{e}_i$ $\mathbf{F}_i\mathbf{r}_i + \mathbf{e}_{i0} = \mathbf{e}_i$ $\mathbf{K}'_i\mathbf{d}'_i = \mathbf{p}'_i$ $\mathbf{T}_i\mathbf{p}_i = \mathbf{p}'_i$ $\mathbf{T}_i\mathbf{d}_i = \mathbf{d}'_i$ $\mathbf{T}_i\mathbf{H}_i = \mathbf{H}'_i$ $\mathbf{T}_i\mathbf{K}_i\mathbf{T}_i^T = \mathbf{K}'_i$			

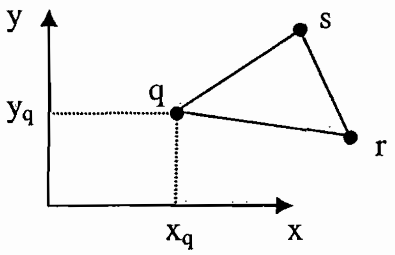
$a = L/AE$, AE = axial stiffness, $u = \cos\alpha$, $v = \sin\alpha$

Basic relationships, for element j :

displacements	$\mathbf{u} = \mathbf{N}^j \mathbf{d}^j$
strains	$\boldsymbol{\varepsilon} = \mathbf{B}^j \mathbf{d}^j$
stresses	$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} = \mathbf{D}\,\mathbf{B}^j \mathbf{d}^j$
stiffness matrix	$\mathbf{K}^j = \int (\mathbf{B}^j)^T \mathbf{D}\,\mathbf{B}^j\, dV$
stiffness equations	$\mathbf{K}^j \mathbf{d}^j = \mathbf{p}^j$

Material stiffness (for plane stress) $\quad \mathbf{D} = \dfrac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$

Shape functions of some simple plane stress elements

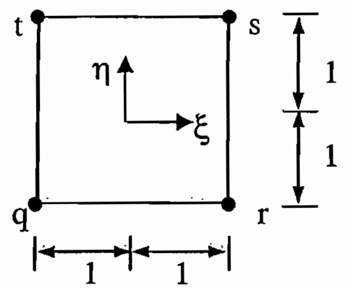


$n_q = [(x_r y_s - x_s y_r) + (y_r - y_s)x + (x_s - x_r)y]/2A$

$n_r = [(x_s y_q - x_q y_s) + (y_s - y_q)x + (x_q - x_s)y]/2A$

$n_s = [(x_q y_r - x_r y_q) + (y_q - y_r)x + (x_r - x_q)y]/2A$

A = area of triangle

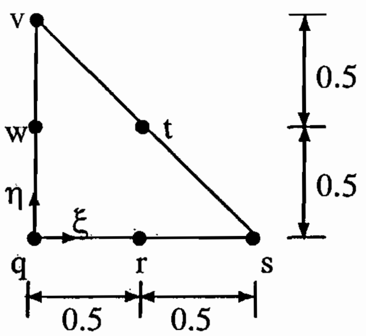


$n_q = (1-\xi)(1-\eta)/4$

$n_r = (1+\xi)(1-\eta)/4$

$n_s = (1+\xi)(1+\eta)/4$

$n_t = (1-\xi)(1+\eta)/4$



$n_q = (1-\xi-\eta)(1-2\xi-2\eta)$

$n_r = 4\xi\,(1-\xi-\eta)$

$n_s = \xi\,(2\xi-1)$

$n_t = 4\,\xi\,\eta$

$n_v = \eta\,(2\eta-1)$

$n_w = 4\eta\,(1-\xi-\eta)$