ENGINEERING TRIPOS PART IIA

Tuesday 29 April 2008 9 to 10.30

Module 3F1

SIGNALS AND SYSTEMS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 Consider the following difference equation:

$$y_{k+1} + a y_k = b u_k, \quad y_0 = \beta$$

where a, b and β are real valued constants.

(a) Take the z-transform of the above difference equation to find $Y(z)$ as a function of $U(z)$ and β. [15%]

(b) With $u_k = 0$ for all $k \geq 0$, find y_k for all $k \geq 0$. Assuming $\beta > 0$, draw graphs of y_k versus k for each of the five cases: (i) $a < -1$, (ii) $-1 < a < 0$, (iii) $a = 0$, (iv) $0 < a < 1$ and (v) $a > 1$. [30%]

(c) Let $\beta = 0$ and consider the feedback law $u_k = -c y_k + r_k$, where c is a real number and r_k is an input to the system.

(i) Show that the transfer function $T(z)$ from $R(z)$ to $Y(z)$ is given by

$$T(z) = \frac{b}{z + a + bc}$$

and write down a necessary and sufficient condition for $T(z)$ to be stable. [20%]

(ii) Assuming that $T(z)$ is stable show that

$$\frac{|b|}{2} \leq \left| T(e^{j\theta}) \right| \leq \frac{|b|}{1 - |a + bc|}$$

for all θ with $0 \leq \theta \leq \pi$. [35%]

2 (a) (i) State the Nyquist stability criterion for a discrete time system $G(z)$ and a constant gain feedback controller k in the usual negative feedback configuration. [10%]

(ii) Sketch the Nyquist diagram for the open-loop system

$$G(z) = \frac{4}{z-2} \quad (1)$$

clearly labelling any intersections with the real axis and the direction of encirclement. [20%]

(iii) Use the Nyquist stability criterion to determine the closed-loop stability of the feedback loop with $G(z)$ given by (1) and with $k = 0.5$ and $k = 2$ in the usual negative feedback configuration. [20%]

(b) The Characteristic Function of a random variable X is defined to be

$$\Phi_X(u) = E[e^{juX}]$$

where $E[.]$ is the expectation operator.

(i) Calculate the Characteristic Function of the random variable X whose probability density function is given by

$$f_X(x) = \begin{cases} ae^{-ax} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

where a is a positive constant. [20%]

(ii) Show that, for any random variable X,

$$\frac{d^n \Phi_X(u)}{du^n} = j^n \int_{-\infty}^{\infty} x^n \, e^{jux} \, f_X(x) \, dx$$

Hence calculate the first and second moments of the random variable X in part (b)(i). [30%]

3 (a) What is the form of the resulting probability density function (pdf) when independent random variables with Gaussian pdfs are added together? Give brief reasons for this result. [20%]

(b) The autocorrelation function (ACF) of a zero-mean ergodic random signal $X(t)$ is the triangular function

$$r_{XX}(\tau) = \begin{cases} A\left(1 - \frac{|\tau|}{T}\right) & \text{for } |\tau| \leq T \\ 0 & \text{for } |\tau| > T \end{cases}$$

where A and T are positive constants.

(i) Calculate the mean power of $X(t)$, assuming it represents the signal voltage across a one ohm resistor. [10%]

(ii) The signal $X(t)$ is the input signal to a linear system whose impulse response is given by

$$h(t) = \delta(t) + \delta(t - T)$$

and whose output is denoted by $Y(t)$. Calculate and sketch the ACF of the output signal, $r_{YY}(\tau)$. Hence obtain the mean power of $Y(t)$. [50%]

(iii) Assuming that $X(t)$ and $X(t-T)$ are independent and Gaussian, give an expression for $f_Y(y)$, the pdf of $Y(t)$. [20%]

4 (a) A message source S has M symbols, with probabilities p_i, $i = 1,\dots,M$. In a Shannon-Fano code the codeword lengths l_i for S are chosen such that

$$\log_2(1/p_i) \leq l_i < \log_2(1/p_i) + 1 \qquad \text{for } i = 1,\dots,M$$

Show that

$$H(S) \leq L < H(S) + 1$$

where $H(S)$ is the entropy of the source and L is the mean code length. [25%]

(b) A first-order Markov source has a joint probability table given by

$P(X_n, X_{n-1})$:

X_n \ X_{n-1}	A	B
A	0.84	0.06
B	0.06	0.04

(i) Determine the equilibrium probabilities, $P(A)$ and $P(B)$, and create a conditional probability table, $P(X_n|X_{n-1})$, for this source. [20%]

(ii) Calculate the conditional entropy $H(X_n|X_{n-1})$ and the mutual information between consecutive symbols $I(X_n;X_{n-1})$, and obtain an expression for the total entropy of N consecutive symbols from this source. [40%]

(iii) Assuming a Shannon-Fano code for a string of N consecutive symbols, give bounds on the mean code length per symbol L/N. [15%]

END OF PAPER