ENGINEERING TRIPOS IIA 2009 **Prof H Hodson**, 1

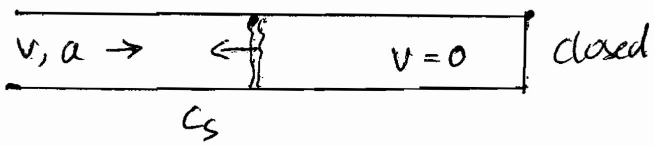
Module 3A3 Examination

1. (a) From Data Book
$$\frac{\rho_s}{\rho} = \frac{(\gamma+1)M^2}{2\left(1+\frac{(\gamma-1)}{2}M^2\right)}$$

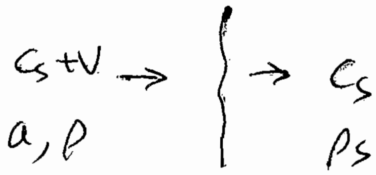
$\div M^2$:
$$\frac{\rho_s}{\rho} = \frac{\gamma+1}{2\left(1/M^2 + \frac{\gamma-1}{2}\right)}$$

Let $M^2 \to \infty$:
$$\underline{\frac{\rho_s}{\rho} \to \frac{\gamma+1}{\gamma-1}}$$
as required.

(b)



Frame of reference of shock wave :



use continuity $\rho_s c_s = \rho(c_s + V)$ and ρ_s/ρ from above.

$$\frac{c_s+V}{c_s} = \frac{(\gamma+1)M_{rel}^2}{2\left(1+\frac{(\gamma-1)}{2}M_{rel}^2\right)}$$
where M_{rel} is the relative Mach number upstream.

$$M_{rel} = \frac{c_s+V}{a} \; ; \quad M = V/a$$

$$\frac{c_s+V}{c_s} = \frac{M_{rel}}{M_{rel}-M}$$

$$\therefore \frac{M_{rel} - M}{M_{rel}} = \frac{2\left(1 + \frac{(\gamma-1)}{2} M_{rel}^2\right)}{(\gamma+1) M_{rel}^2}$$

$$M_{rel}^2 - \frac{(\gamma+1)}{2} M M_{rel} - 1 = 0$$

$$\underline{M_{rel} = \frac{(\gamma+1)}{4} M + \sqrt{1 + \left(\frac{\gamma+1}{4} M\right)^2}}$$ [negative root not physical]

(c) 400 K, 300 ms^{-1}, 101 kPa.

speed of sound $a = \sqrt{\gamma R T} = \sqrt{1{\cdot}4 \times 287 \times 400} = 400{\cdot}9 \text{ ms}^{-1}$

$\therefore$ Mach number $M = \frac{300}{400{\cdot}9} = 0{\cdot}748$

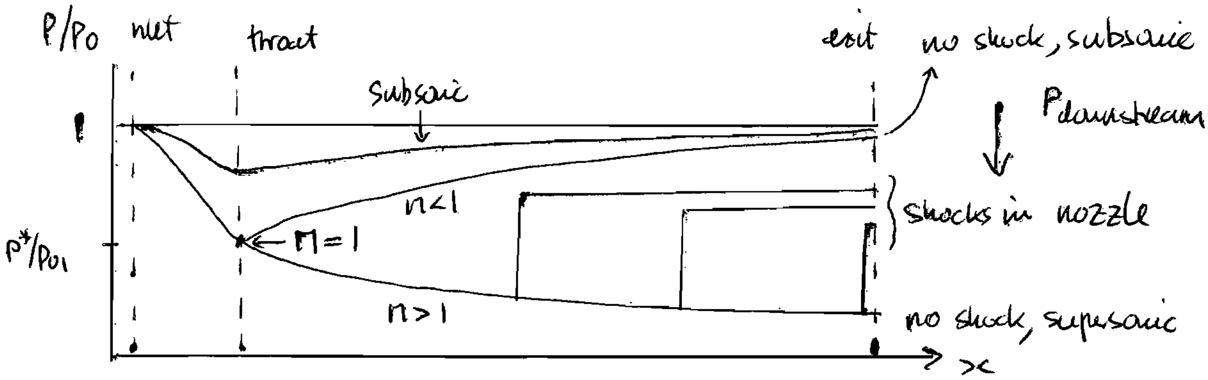
$$\therefore \frac{\gamma+1}{4} M = 0{\cdot}6 \times 0{\cdot}748 = 0{\cdot}4488$$

$$\therefore M_{rel} = 0{\cdot}4488 + \sqrt{1 + (0{\cdot}4488)^2} = 1{\cdot}545$$

tables at $M_{rel} = 1{\cdot}55$: $P_s/p = 2{\cdot}6363$

$$\therefore P_s = 2{\cdot}6363 \times 1{\cdot}01 = \underline{2{\cdot}663 \text{ kPa}}$$

2(a)



(b) With the throat choked $\dfrac{\dot m\sqrt{c_pT_{01}}}{A_t P_{01}} = 1\cdot281$ (tables)

$$A \propto d^2 ; \quad d \propto x^{1/2} \quad \therefore \quad A \propto x$$

$$\frac{d_e}{d_t} = 1\cdot25 \quad \therefore \quad \frac{A_e}{A_t} = 1\cdot25^2 = 1\cdot5625$$

$$\therefore \frac{\dot m\sqrt{c_pT_{01}}}{A_e P_{01}} = \frac{1\cdot281}{1\cdot5625} = 0\cdot81984$$

tables $M = 0\cdot41$ sub
$\underline{M = 1\cdot91}$ super

(i) $\therefore$ tables : $\underline{P_e/P_{01} = 0\cdot147}$ for no shock

(ii) Shock just ahead of exit : $M = 1\cdot91$

tables $P_s/P_{01} = P_s/P \times P/P_{01} = 4\cdot0895 \times 0\cdot147$

$$= \underline{0\cdot601}$$

(c) $\dfrac{P_{01} - P_{0e}}{P_{01}} = 0\cdot15$; $\therefore \dfrac{P_{0e}}{P_{01}} = 0\cdot85$

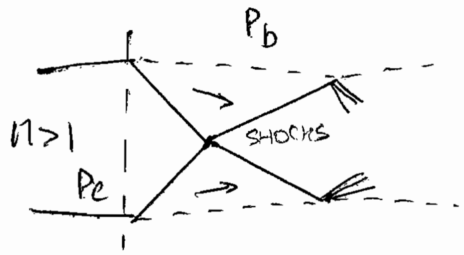
tables $M = 1.71$, $\dfrac{\dot{m}\sqrt{c_p T_{01}}}{A P_{01}} = 0.9509$

$$\frac{A}{A_t} = \frac{1.281}{0.9509} = 1.347 = \left(\frac{d}{d_t}\right)^2 \quad \therefore \frac{d}{d_t} = 1.16$$

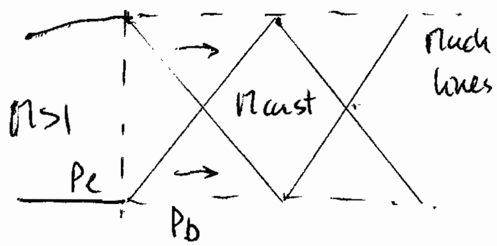
Now $\dfrac{d - d_t}{d_e - d_t} = \left(\dfrac{x - x_t}{x_e - x_t}\right)^{1/2} \quad \therefore \quad \left(\dfrac{0.16}{0.25}\right)^2 = \dfrac{x - x_t}{x_e - x_t}$

ie Shock is located at 41% of the distance <u>between the throat and the exit of the nozzle</u>.

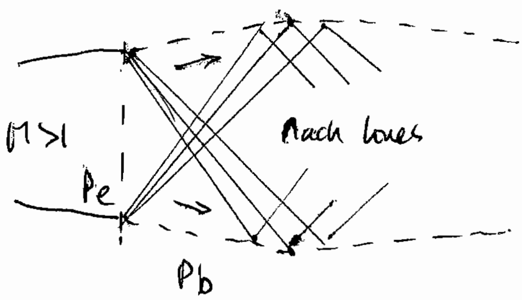
(d)



over expanded $P_e < P_b$

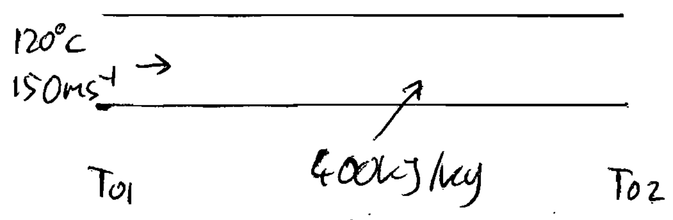


perfectly expanded $P_e = P_b$



under expanded $P_e > P_b$

3



Rayleigh line process. Speed of sound $a = \sqrt{\gamma R T}$

$= \sqrt{1.4 \times 287 \times (273.15 + 120)}$

$= 397.5\,ms^{-1}$

$\therefore M_1 = 0.3774$

$$T_{01}/T_1 = \left(1 + \frac{(\gamma-1)}{2} M^2\right) = 1.0285$$

$\therefore T_{01} = 404.35K$

(a) Rise in stagnation temperature $= \dfrac{400kJ/kg}{1.005kJ/kgK} = 398K$

$\therefore T_{02} = 404.35 + 398 = 802.35K$

tables:
$$\frac{\dfrac{F_2}{\dot{m}\sqrt{c_p T_{02}}}}{\dfrac{F_1}{\dot{m}\sqrt{c_p T_{01}}}} = \left(\frac{404.35}{802.35}\right)^{1/2} = 0.7099$$

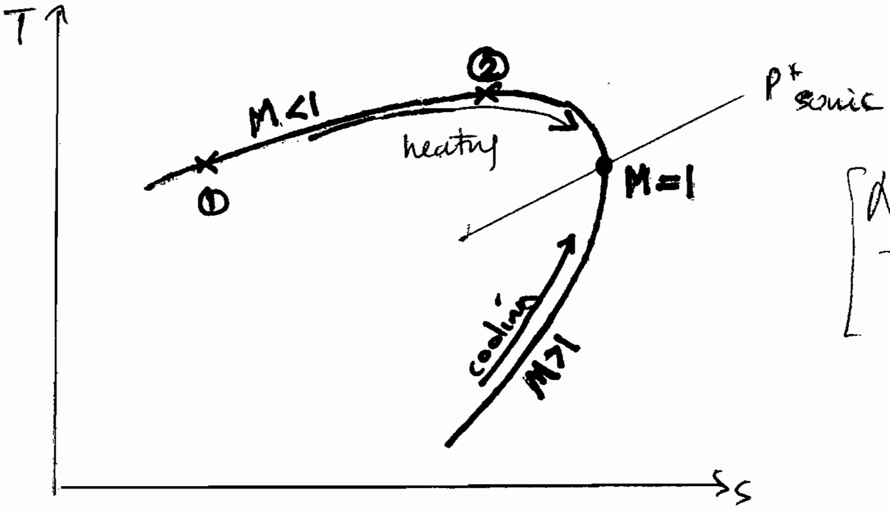
At entry $M_1 = 0.38$; $\dfrac{F_1}{\dot{m}\sqrt{c_p T_{01}}} = 1.409$ (tables)

$$\therefore \frac{F_2}{\dot{m}\sqrt{c_p T_{02}}} = 1.409 \times 0.7099 = 1.0002491$$

tables $\underline{M_2 = 0.84}$ at exit

$(\simeq 1/\sqrt{8})$

(b)



[Note: T_{MAX} @ $M = 1/\sqrt{\gamma}$
$= 0.845$]

(c) Increased heating: pipe exit chokes.

tables for $M_2 = 1$: $\dfrac{F_2}{\dot{m}\sqrt{c_pT_{02}}} = 0.9897$ [minimum value of $F/\sqrt{c_pT_0}$]

$$\frac{T_{02}}{T_{01}} = \left(\frac{\dfrac{F_1}{\dot{m}\sqrt{c_pT_{01}}}}{\dfrac{F_2}{\dot{m}\sqrt{c_pT_{02}}}} \right)^2 = \left(\frac{1.409}{0.9897}\right)^2 = 2.0268$$

$$\therefore T_{02} = 2.0268 \times 404.35 = 819.54\text{K}$$

$$\therefore \Delta T_0 = 819.54 - 404.35 = 415.19\text{K}$$

$$\therefore Q = c_p \Delta T_0 = 1005 \times 415.19 = \underline{417.3 \text{ kJ/kg}}$$

This is a small increase over the original 400 kJ/kg – only 4.3%. Nevertheless it increases the exit Mach number from 0.84 to 1.0.

4 a) i) mass: $\nabla.(\rho V)=0 \Rightarrow \nabla.V + \frac{1}{\rho} V.\nabla\rho = 0$

ii) energy: $h_0 = const = c_p T + \frac{(u^2+v^2)}{2} = \frac{a^2}{\gamma-1} + \frac{1}{2}(u^2+v^2)$

iii) momentum: $-\frac{1}{\rho}\nabla p = V.\Delta V$

b) flow is isentropic $\Rightarrow p = k\rho^\gamma \Rightarrow dp = k\gamma\rho^{\gamma-1}d\rho \Rightarrow \nabla p = a^2\nabla\rho$

$$\therefore \nabla.V = -\frac{1}{\rho}V.\nabla\rho = \frac{-1}{\rho a^2}V.\nabla p = \frac{1}{a^2}V.(V.\nabla V)$$

thus $a^2\nabla.V - V(V.\nabla V) = 0$

or consider velocity components u, v in x, y directions respectively:

$$a^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - [u\underline{e_x} + v\underline{e_y}].\left[\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)\underline{e_x} + \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)\underline{e_y}\right] = 0$$

$$a^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - u\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) - v\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = 0$$

$$(a^2-u^2)\frac{\partial u}{\partial x} - uv\frac{\partial u}{\partial y} - vu\frac{\partial v}{\partial x} + (a^2-v^2)\frac{\partial v}{\partial y} = 0$$

Irrotational flow $\therefore \nabla\times V = 0 \Rightarrow \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

$$\therefore (a^2-u^2)\frac{\partial u}{\partial x} - 2uv\frac{\partial u}{\partial y} + (a^2-v^2)\frac{\partial v}{\partial y} = 0$$

c) Consider a small perturbation to uniform on-coming x-wise velocity U_∞ in terms of velocity potential ϕ

hence subs: $U_\infty + \frac{\partial\phi}{\partial x}$ for u & $\frac{\partial\phi}{\partial y}$ for v above

thus: $$\left[a^2 - \left(U_\infty + \frac{\partial\phi}{\partial x}\right)^2\right]\frac{\partial^2\phi}{\partial x^2} - 2\left[U_\infty + \frac{\partial\phi}{\partial x}\right]\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y} + \left[a^2 - \left(\frac{\partial\phi}{\partial y}\right)^2\right]\frac{\partial^2\phi}{\partial y^2} = 0$$

Since perturbation & hence $\frac{\partial\phi}{\partial x}$ & $\frac{\partial\phi}{\partial y}$ are small,

neglect 2nd & H.O.T.; and divide through by a^2

thus: $$\left(1 - M_\infty^2\right)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

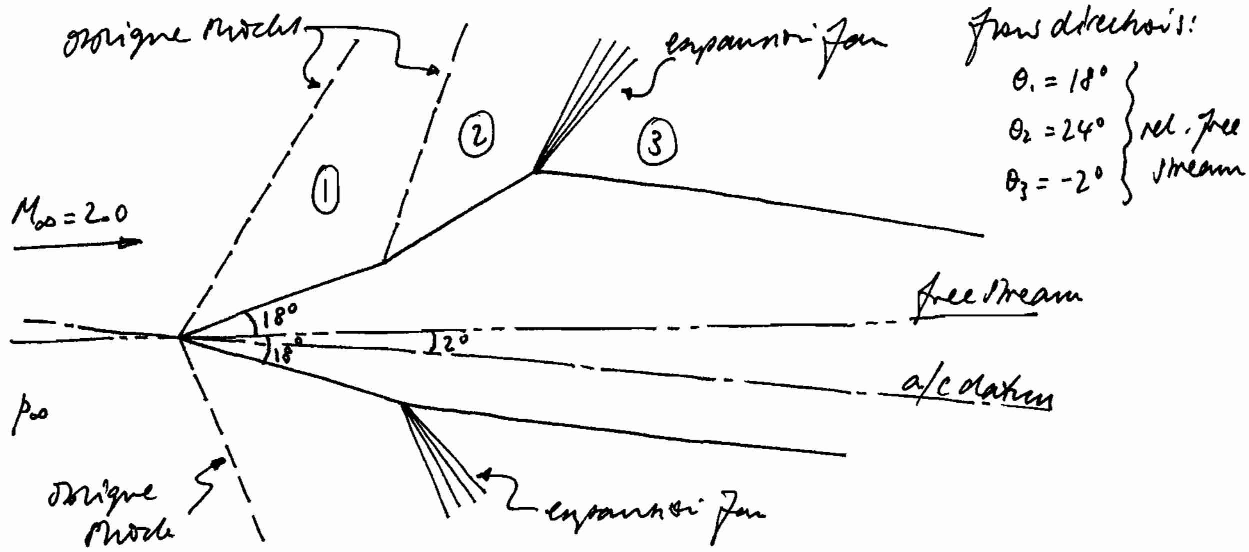
d) From the above it follows that $C_m = \frac{C_{m_0}}{\sqrt{1-M_\infty^2}}$ (Prandtl-Glauert)

control authority limit when $C_{m_{wing}} = 6\ \text{kNm} = |C_{m_{tail}}|$

i.e. when $\frac{C_m}{C_{m_0}} = 1.2$

thus limiting M_∞ given by: $\frac{1}{\sqrt{1-M_\infty^2}} = 1.2 \Rightarrow M_\infty = 0.55$

5 a) Check: @ $M_\infty = 2.0$ max $\delta = 22.925°$ from tables.



b) for $\delta = 18°$, @ $M = 2.0$ take weak solution from tables:

$$\beta = 49.833°, \quad \frac{p_1}{p_\infty} = 2.556, \quad M_1 = 1.311$$

from ① → ② $\delta = 6°$. interpolation gives:

$$\frac{p_2}{p_1} = 1.404, \quad M_2 = 1.045 \Rightarrow p_1 = 2.556\,p_\infty \;\therefore\; p_2 = 3.589\,p_\infty$$

from ② → ③ $\nu + \theta$ = const across the char

$$M_2 = 1.045 \Rightarrow \nu_2 = 0.42, \quad \theta_2 = 24°, \quad \theta_3 = -2°$$

$$\Rightarrow \nu_3 = 26.42° \quad \therefore\; M_3 = 2.004 \text{ for which } \frac{p_0}{p} = 7.875$$

@ $M_2 = 1.045$, $\frac{p_0}{p} = 1.999 \Rightarrow p_{0_2} = 7.174\,p_\infty$

isentropic expansion fan i.e. $p_{0_3} = p_{0_2}$

$$\Rightarrow p_3 = \frac{7.174}{7.875}\,p_\infty = 0.911\,p_\infty$$

cockpit pressure $= 2\,p_\infty \Rightarrow$ pressure ratio across canopy $= \frac{2}{0.911} = 2.195$ i.e. +10% over stationary.

c) Check nose shock: max δ @ $M_\infty = 1.80 = 19.145°$

$\Rightarrow$ oblique shocks remain attached on both upper & lower surfaces of nose & $M_1 = 1.074$

<u>Lower surface</u>

New angles, but flowfield remains essentially unchanged w. attached nose plane oblique shock followed by expansion fan @ nose/belly junction.

<u>Upper surface</u>

Attached oblique plane shock on vertex of nose.

max δ @ $M_1 = 1.074 = 1.016°$

$\therefore$ attached plane oblique shock cannot now form in windscreen/nose junction & a <u>normal</u> shock will replace it.

This may result in shock-induced flow separation @ base of windscreen.

New $M_2 = 0.978$ i.e. subsonic

$\Rightarrow$ expansion fan cannot form @ windscreen/canopy junction $\therefore$ if not already separated, separation highly likely here.

d) Curvature of conical nose (orthogonal to free stream direction) effectively reduces back pressure on shock & weakens it hence, all other things being equal, expect lower wave drag than 2-D analysis suggests.

6 (a) $$\phi^{n+1} = \phi^n + \Delta t \left.\frac{\partial \phi}{\partial t}\right|^n + \frac{\Delta t^2}{2!} \left.\frac{\partial^2 \phi}{\partial t^2}\right|^n + O(\Delta t^3)$$

$$F^{n-1} = F^n - \Delta t \left.\frac{\partial F}{\partial t}\right|^n + \frac{\Delta t^2}{2!} \left.\frac{\partial^2 F}{\partial t^2}\right|^n + O(\Delta t^3)$$

(b) $$\left.\frac{\partial \phi}{\partial t}\right|^n = -F^n \quad ; \quad \left.\frac{\partial^2 \phi}{\partial t^2}\right|^n = - \left.\frac{\partial F}{\partial t}\right|^n$$

thus: $$\phi^{n+1} = \phi^n - \Delta t\, F^n - \frac{\Delta t^2}{2!} \left.\frac{\partial F}{\partial t}\right|^n + O(\Delta t^3) \qquad ①$$

Now $$\Delta t(aF^n + bF^{n-1}) = \Delta t(a+b) F^n - \Delta t^2 b \left.\frac{\partial F}{\partial t}\right|^n + O(\Delta t^3)$$

$$\therefore \phi^{n+1} = \phi^n - \Delta t (a+b) F^n + \Delta t^2 b \left.\frac{\partial F}{\partial t}\right|^n + O(\Delta t^3) \qquad ②$$

We require ① and ② to be equal to second order.

$\therefore$ $a+b=1$ and $\underline{b = -1/2}$ $\therefore$ $\underline{a = 3/2}$

(c) For $a=2$ and $b=-1$ we have only first order, since $a+b=1$ but $b \neq -1/2$.

Have
$$\left.\begin{aligned} \phi^{n+1} &= \phi^n + \Delta t \left.\frac{\partial \phi}{\partial t}\right|^n + \frac{\Delta t^2}{2!} \left.\frac{\partial^2 \phi}{\partial t^2}\right|^n + O(\Delta t^3) \\ \phi^{n+1} &= \phi^n - \Delta t(a+b) F^n + \Delta t^2 b \left.\frac{\partial F}{\partial t}\right|^n + O(\Delta t^3) \end{aligned}\right\}$$

Hence, on subtracting:
$$\left.\frac{\partial \phi}{\partial t}\right|^n + (a+b) F^n = \Delta t \left[-\frac{1}{2} \left.\frac{\partial^2 \phi}{\partial t^2}\right|^n + b \left.\frac{\partial F}{\partial t}\right|^n \right] + O(\Delta t^3)$$

but $\alpha + b = 2 - 1 = 1$ and $b = -1$

$$\therefore \left.\frac{\partial \phi}{\partial t}\right|^n + F^n = -\Delta t \left[\frac{1}{2} \left.\frac{\partial^2 \phi}{\partial t^2}\right|^n + \left.\frac{\partial F}{\partial t}\right|^n \right] + O(\Delta t^3)$$

Now $\quad \frac{\partial \phi}{\partial t} = -F = -A \frac{\partial \phi}{\partial x} \quad$ (scalar convection equation)

$$\therefore \frac{\partial F}{\partial t} = A \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial t}\right) = -A^2 \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial t^2}$$

$$\therefore \frac{\partial \phi}{\partial t} + F = \Delta t A^2 \frac{\partial^2 \phi}{\partial x^2}\left(1 - \frac{1}{2}\right) = \frac{\Delta t A^2}{2} \frac{\partial^2 \phi}{\partial x^2}$$

$\therefore$ Discrete equation has a source term with a second spatial derivative : diffusion term.

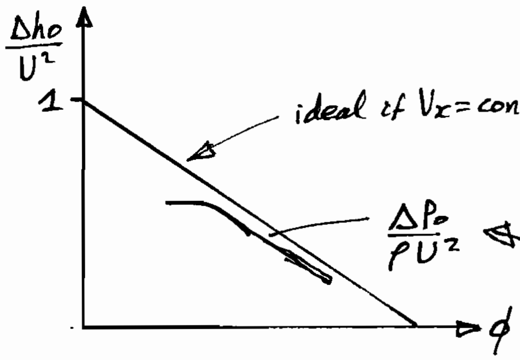
This does not exist in the original pde.

Possible benefit in terms of stability by smoothing steep gradients in the solution.

Full Question

7a)
$$\Delta h_0 = U_2 V_{\theta 2} - U_1 V_{\theta 1} \qquad U_2 = U_1 = U \qquad V_{x1} = V_{x2} = V_x$$
$$= U(V_{\theta 2,rel} + U) - U V_{\theta 1}$$
$$= U^2 + U V_x \tan\alpha_{2,rel} - U V_x \tan\alpha_1$$
$$\Rightarrow \underline{\underline{\frac{\Delta h_0}{U^2} = 1 - \phi\left(\tan\alpha_{2,rel} - \tan\alpha_1\right)}} \qquad \phi = \frac{V_x}{U}$$

b)



$\alpha_1 = 0$

$\Rightarrow \frac{\Delta h_0}{U^2} = 1 + \phi \tan\alpha_{2,rel}$

N.B. $\alpha_{2,rel} < 0$

Difference due to irreversibilities. $(\eta < 1)$

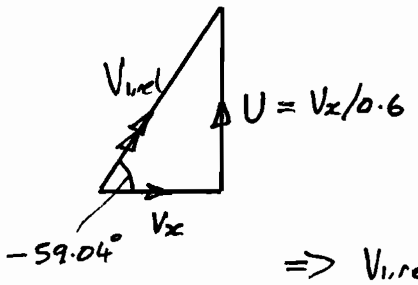
Note axial velocity will only be constant at design or for low mach number compressors.

c) $\tan\alpha_{1,rel} = \tan\alpha_1 - \frac{1}{\phi} = 0.0 - \frac{1}{0.6}$

$$\underline{\underline{\alpha_{1,rel} = -59.04^\circ}}$$

$$\frac{\Delta h_0}{U^2} = 1 + \phi \tan\alpha_{2,rel} \quad \Rightarrow 0.4 = 1 + 0.6 \tan\alpha_{2,rel}$$
$$\Rightarrow \tan\alpha_{2,rel} = -1.0$$
$$\Rightarrow \underline{\underline{\alpha_{2,rel} = -45^\circ}}$$

d)



$$V_{1,rel} = \sqrt{V_x^2 + U^2} = V_x\sqrt{1 + 1/0.6^2}$$
$$\Rightarrow \frac{V_{1,rel}}{a_1} = \frac{V_x}{a_1}\sqrt{1 + \frac{1}{0.6^2}}$$
$$\Rightarrow \frac{V_{1,rel}}{a_1} = 0.4\sqrt{1 + \frac{1}{0.6^2}} = \underline{0.7775 = M_{1,rel}}$$

TABLES $M = 0.7775 \Rightarrow \underline{\underline{\frac{V_{1,rel}}{\sqrt{c_p T_{01,rel}}} = 0.4645}}$

e) Fixed radius $\Rightarrow T_{02,rel} = T_{01,rel}$

Constant axial velocity $V_{x2} = V_{x1} = 1$

$$V_x = V_{1,rel}\cos\alpha_{1,rel} = V_{2,rel}\cos\alpha_{2,rel}$$
$$\Rightarrow V_{2,rel} = V_{1,rel}\frac{\cos\alpha_{1,rel}}{\cos\alpha_{2,rel}}$$

$$\frac{V_{2,rel}}{\sqrt{c_p T_{02,rel}}} = \frac{V_{1,rel}}{\sqrt{c_p T_{01,rel}}} \times \frac{\cos\alpha_{1,rel}}{\cos\alpha_{2,rel}} = \frac{0.4645\cos(-59.04)}{\cos(-45^\circ)} = \underline{\underline{0.3379}}$$

FULL QUESTION CONT.

7f)
$$Y_p = \frac{P_{01,rel} - P_{02,rel}}{P_{01,rel} - P_1}$$

$$\Rightarrow Y_p = \frac{1 - P_{02,rel}/P_{01,rel}}{1 - P_1/P_{01,rel}} \Rightarrow \frac{P_{02,rel}}{P_{01,rel}} = 1 - Y_p\left(1 - \frac{P_1}{P_{01,rel}}\right)$$

$$\Rightarrow \frac{P_{02,rel}}{P_{01,rel}} = 1 - 0.05\left(1 - \frac{1}{\left(1+\frac{\gamma-1}{2}0.7775^2\right)^{3.5}}\right)$$

$$\frac{P_{02,rel}}{P_{01,rel}} = 1 - 0.05\left(1 - 0.67068\right) = \underline{0.98353}$$

$$\frac{\dot{m}\sqrt{C_p T_{01,rel}}}{P_{01,rel}\,A_{x1}\cos\alpha_{1,rel}} = Q_t(M_{1,rel}) \qquad \frac{\dot{m}\sqrt{C_p T_{02,rel}}}{P_{02,rel}\,A_{x2}\cos\alpha_{2,rel}} = Q_t(M_{2,rel})$$

$$\frac{V_{2,rel}}{\sqrt{C_p T_{02,rel}}} = 0.3379 \Rightarrow \underline{M_{2,rel} = 0.5502}$$

$$\frac{A_{x2}}{A_{x1}} = \frac{Q_t(M_{1,rel})}{Q_t(M_{2,rel})}\;\frac{P_{01,rel}}{P_{02,rel}}\;\frac{\cos\alpha_{1,rel}}{\cos\alpha_{2,rel}}$$

$$= \frac{1.2220}{1.0210}\;\frac{1}{0.98353}\;\frac{\cos(-59.04^\circ)}{\cos(-45^\circ)}$$

$$\underline{\underline{\frac{A_{x2}}{A_{x1}} = 0.8853}}$$

(12% BLADE HEIGHT REDUCTION)

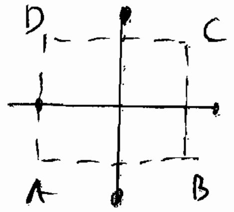
g)
$$\frac{\dot{m}\sqrt{C_p T_{01}}}{P_1\,A_{x1}\cos\alpha_{1,rel}} = Q_s(M_{1,rel})$$

$$\Rightarrow \frac{P_2}{P_1} = \frac{Q_s(M_{1,rel})}{Q_s(M_{2,rel})}\;\frac{A_{x1}}{A_{x2}}\;\frac{\cos\alpha_{1,rel}}{\cos\alpha_{2,rel}}$$

$$= \frac{1.8221}{1.2543}\;\frac{1}{0.8853}\;\frac{\cos(-59.04)}{\cos(-45^\circ)}$$

$$\underline{\underline{\frac{P_2}{P_1} = 1.194}}$$

8 (a) (i)



$$\nabla^2 \phi = 0$$

Gauss Theorem in 2D

$$\int \nabla^2 \phi \, dA = \oint \nabla \phi \cdot \underline{n} \, dl = 0$$

Take $\underline{n}$ as the outward-pointing normal vector.

Evaluate line integrals : uniform Cartesian mesh !

$$AB: \nabla \phi \cdot \underline{n} \, dl = \frac{\phi_{i,j-1} - \phi_{i,j}}{\Delta y} \Delta x$$

$$BC: \nabla \phi \cdot \underline{n} \, dl = \frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta x} \Delta y$$

$$CD: \nabla \phi \cdot \underline{n} \, dl = \frac{\phi_{i,j+1} - \phi_{i,j}}{\Delta y} \Delta x$$

$$DA: \nabla \phi \cdot \underline{n} \, dl = \frac{\phi_{i-1,j} - \phi_{i,j}}{\Delta x} \Delta y$$

Add and divide through by $\Delta x \Delta y$:

$$\frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{\Delta x^2} + \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{\Delta y^2} = 0$$

(ii) Taylor series in x :

$$\phi_{i+1,j} = \phi_{i,j} + \left.\frac{\partial \phi}{\partial x}\right|_{i,j} \Delta x + \left.\frac{\partial^2 \phi}{\partial x^2}\right|_{i,j} \frac{\Delta x^2}{2!} + \left.\frac{\partial^3 \phi}{\partial x^3}\right|_{i,j} \frac{\Delta x^3}{3!} + \left.\frac{\partial^4 \phi}{\partial x^4}\right|_{i,j} \frac{\Delta x^4}{4!} + O(\Delta x^5)$$

Repeat for $\phi_{i-1,j}$ and add:

$$\phi_{i+1,j} + \phi_{i-1,j} = 2\phi_{i,j} + \left.\frac{\partial^2\phi}{\partial x^2}\right|_{i,j}\Delta x^2 + 2\left.\frac{\partial^4\phi}{\partial x^4}\right|_{i,j}\frac{\Delta x^4}{4!} + O(\Delta x^6)$$

$$\therefore \quad \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{\Delta x^2} = \left.\frac{\partial^2\phi}{\partial x^2}\right|_{i,j} + O(\Delta x^2)$$

and similarly for $\frac{\partial^2\phi}{\partial y^2}$. QED.

HALF QUESTION

8b) i) ASSUME : NO RADIAL VELOCITY (PARALLEL STREAMLINES IN MERIDIONAL VIEW)
UNIFORM ENTROPY (INVISCID, ISENTROPIC FLOW)
STEADY (NO TIME DEPENDENCE)

FLOW WITH CIRCULAR STREAMLINES $\frac{dP}{dr} = \rho \frac{V_\theta^2}{r}$

$$T\frac{ds}{dr} = \frac{dh_o}{dr} - \frac{1}{\rho}\frac{dP}{dr} \qquad \frac{ds}{dr} \equiv 0$$

$$\Rightarrow \quad \frac{dh}{dr} = \frac{V_\theta^2}{r}$$

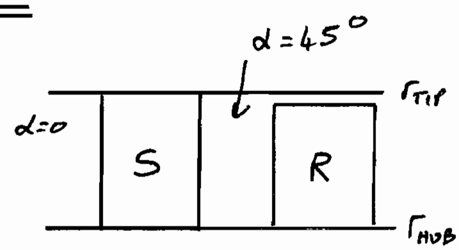
$$h_o = h + \tfrac{1}{2}(V_x^2 + V_\theta^2)$$

$$\Rightarrow \frac{dh_o}{dr} = \frac{dh}{dr} + V_x\frac{dV_x}{dr} + V_\theta\frac{dV_\theta}{dr}$$

$$\Rightarrow \frac{dh_o}{dr} = V_x\frac{dV_x}{dr} + \frac{V_\theta^2}{r} + V_\theta\frac{dV_\theta}{dr}$$

$$\Rightarrow \underline{\underline{\frac{dh_o}{dr} = V_x\frac{dV_x}{dr} + \frac{V_\theta}{r}\frac{d}{dr}(rV_\theta)}}$$

ii) $V_\theta = V_x \tan\alpha$ AT STATOR EXIT



UNIFORM INLET FLOW, NO SWIRL $\Rightarrow \left.\frac{dh_o}{dr}\right|_{inlet} = 0$

STATOR DOES NO WORK $\Rightarrow \left.\frac{dh_o}{dr}\right|_{\substack{STATOR\\EXIT}} = 0$

$$\Rightarrow \quad 0 = V_x\frac{dV_x}{dr} + \frac{V_x\tan\alpha}{r}\frac{d}{dr}(rV_x\tan\alpha)$$

$$0 = V_x\frac{dV_x}{dr} + \frac{V_x^2\tan^2\alpha}{r} + \tan^2\alpha\, V_x\frac{dV_x}{dr}$$

$$0 = V_x\left(1+\tan^2\alpha\right)\frac{dV_x}{dr} + \frac{V_x^2\tan^2\alpha}{r}$$

$$\Rightarrow 0 = \frac{dV_x}{dr} + \frac{V_x\sin^2\alpha}{r}$$

$$\int_{HUB}^{TIP}\frac{1}{V_x}\,dV_x = -\sin^2\alpha\int_{HUB}^{TIP}\frac{1}{r}\,dr$$

$$\ln\left(\frac{V_{x,TIP}}{V_{x,HUB}}\right) = -\sin^2\alpha\,\ln\left(\frac{r_{TIP}}{r_{HUB}}\right)$$

$$\Rightarrow \frac{V_{x,TIP}}{V_{x,HUB}} = \left(\frac{r_{HUB}}{r_{TIP}}\right)^{\sin^2\alpha} \qquad \alpha = 45^\circ,\ r_{HUB}/r_{TIP} = 0.8$$

$$\Rightarrow \underline{\underline{\frac{V_{x,TIP}}{V_{x,HUB}} = \sqrt{0.8} = 0.894}}$$

NEED RADIALLY INWARD PRESSURE GRADIENT
$\Rightarrow V_{x,TIP} < V_{x,HUB}$.

ENGINEERING TRIPOS PART IIA 2009

ANSWRES: MODULE 3A3: FLUID MECHANICS II

Q1:	2.6 kPa		
Q2:	(b)	0.147, 0.601	
	(c)	41% of distance from throat to exit of nozzle	
Q3	(a)	0.378	0.845
	(c)	417 kJ/kg	
Q5	(b)	2.2	
Q6	(b)	a=1.5	b= -0.5
Q7	(c)	-45 deg	-59 deg
	(e)	0.338	
	(f)	0.885	
	(g)	1.194	