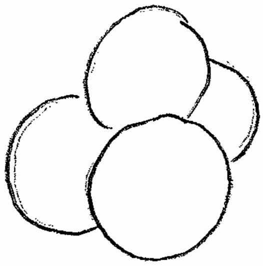
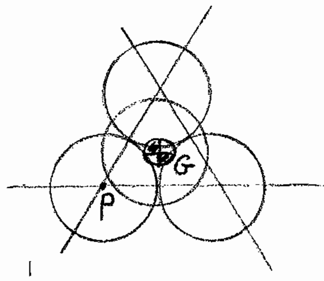
Prof R S Langley

3C5 - DYNAMICS : CRIBS 2009

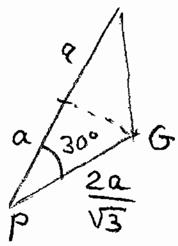
1.(a) The ornament has four axes that are evidently principal and since there are only three principal axes for an "ABC" body the ornament must be "AAA".





Find "A" using a top view

$$A = \underbrace{4 \cdot \tfrac{2}{5} m a^2}_{\text{4 spheres}} + \underbrace{3\, m \left(\tfrac{2a}{\sqrt{3}}\right)^2}_{\text{parallel axis theorem for the 3 "base" spheres}}$$

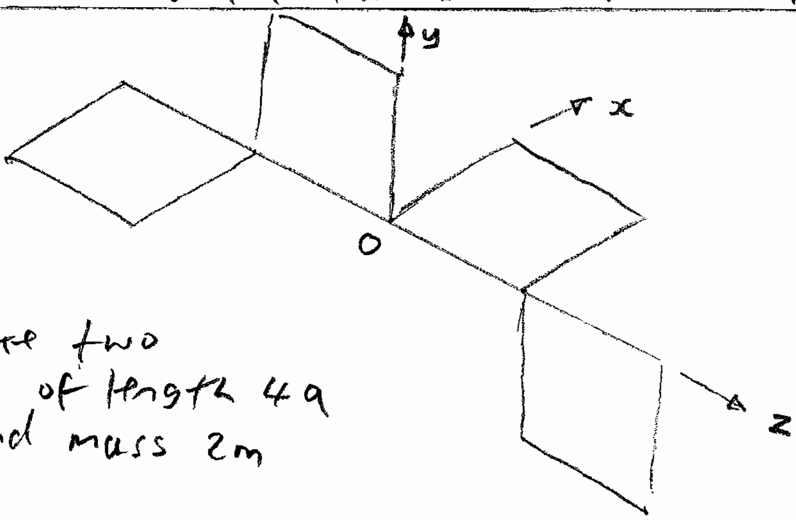


[40%]

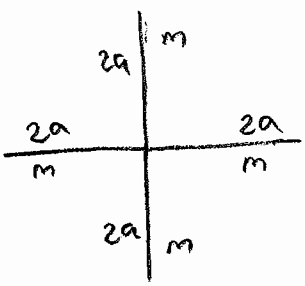
$$= \underline{\underline{\frac{28}{5} m a^2}}$$

and all axes are principal with this same value A

(b) (i)



end view (along z) looks like two rods of length 4a and mass 2m



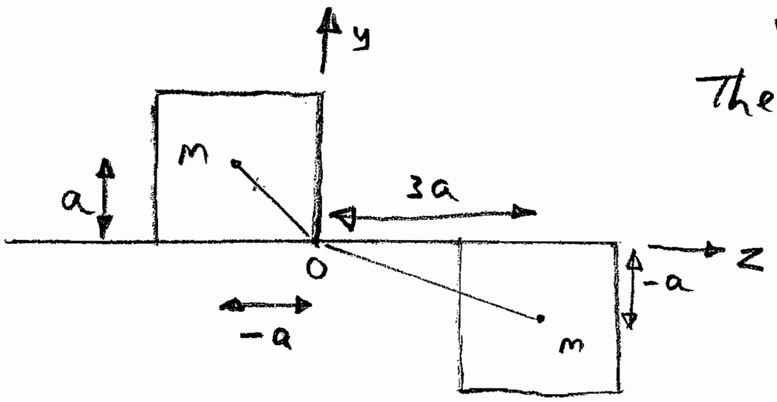
$$\therefore I_{zz} = 2 \times \frac{1}{12}(2m)(4a)^2$$

$$= \frac{16}{3} m a^2$$

[20%]

1 cont (b)(ii) For I_{xy} use z-view again and note that all mass is on line $x=0$ or $y=0$ $\therefore$ $I_{xy} = \int xy\,dm = 0$ [20%]

(c)(iii) y-z view



Only two plates contribute.
The plates themselves have $I_{yz} = 0$ at their centres

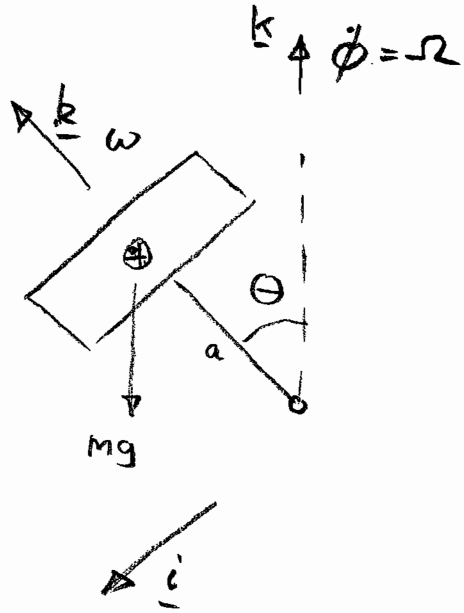
So only need to use parallel axes theorem

$$I_{yz} = m(-a)(a) + m(3a)(-a)$$

$$= -4ma^2$$ [20%]

2.(a) Gyroscope equations

$$A\dot{\Omega}_1 - (A\Omega_3 - C\omega_3)\Omega_2 = Q_1$$
$$A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1 = Q_2$$
$$C\dot{\omega}_3 = Q_3$$



with $\Omega_1 = -\dot{\phi}\sin\theta$

$\Omega_2 = \dot{\theta}$

$\Omega_3 = \dot{\phi}\cos\theta$

Couples $Q_1 = 0$ $\quad Q_2 = mga\sin\theta$ $\quad Q_3 = 0$

Gyro equation (2) in steady state $\dot{\Omega}_2 = 0$
and fast spin $\omega_3 \gg \Omega_3$

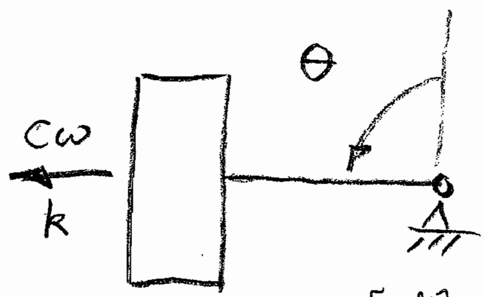
$$\therefore \; -C\omega_3(-\dot{\phi}\sin\theta) = mga\sin\theta$$

$$\therefore \; \dot{\phi} = \frac{mga}{C\omega_3} \quad \text{independent of } \theta \qquad [50\%]$$

(b) (i) $\underset{\sim}{Q} = \dot{\underline{h}}$ so if couple $= Q\underline{K}$

then $\dot{\underline{h}} = Q\underline{K}$ also

For fast spin the only important component of $\underline{h}$ is $\underset{\sim}{h} \simeq C\omega\,\underline{k}$

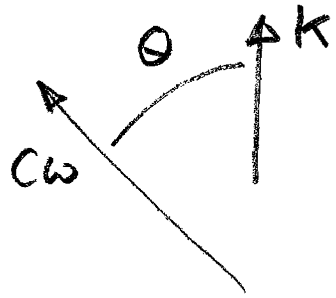


[20%]

2 cont (b) and spin ω is constant

so if $\underline{h}$ is to change it can only change in direction.

Component of $\underline{h}$ in $\underline{k}$ direction



$$\underline{h} \cdot \underline{k} = C\omega \cos\theta$$

rate of change $\frac{d}{dt}(C\omega\cos\theta) = -\dot{\theta}C\omega\sin\theta$

and for $\theta = \frac{\pi}{2}$ $\quad \frac{d}{dt}(\underline{h}\cdot\underline{k}) = -C\omega\dot{\theta}$

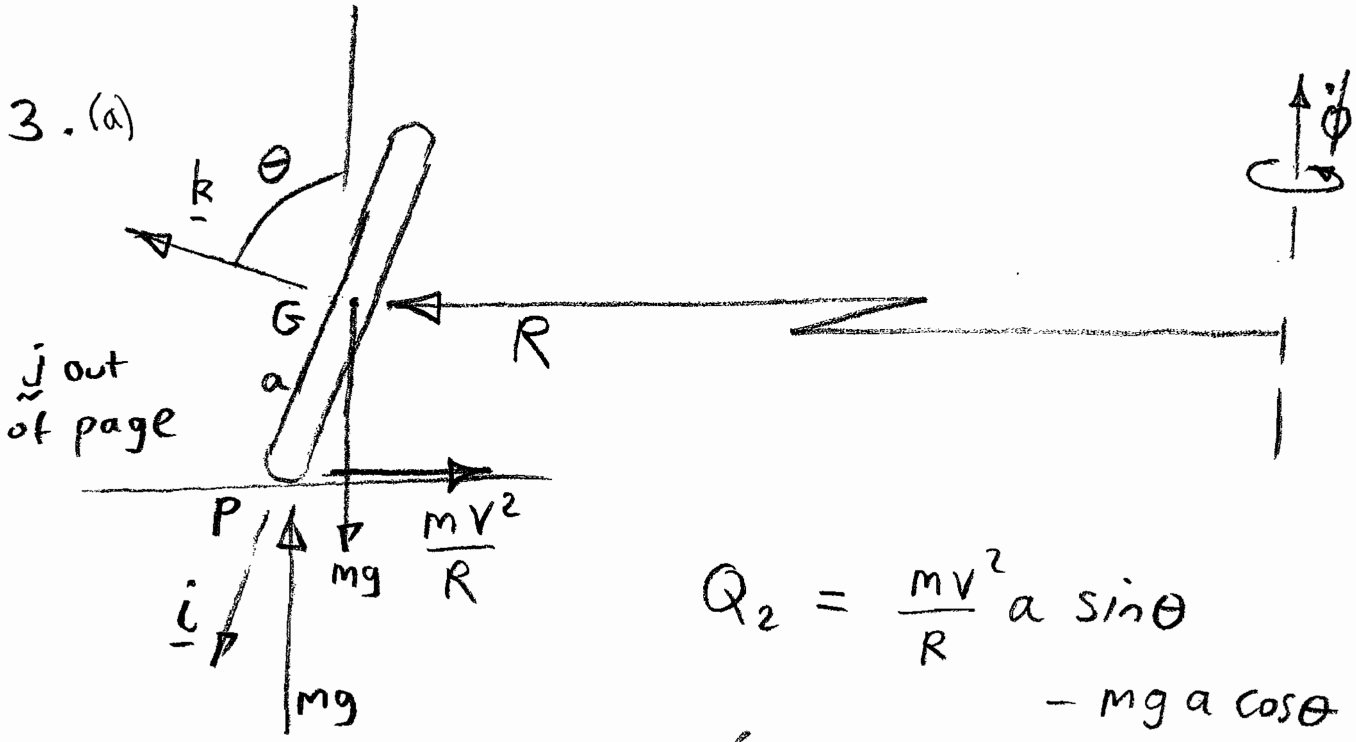
and with $\underline{\dot{h}} = Q\underline{k}$

obtain $\dot{\theta} = \dfrac{-Q}{C\omega}$ [30%]

NB/ friction would produce a negative Q, and so $\dot{\theta}$ would be positive.

In the exam, many students considered the first gyro equation, rather than the method outlined above. This gives the same answer for $\dot{\theta}$ when $\theta = \pi/2$, but an incorrect answer at other values of θ. Why?

3. (a)



$$Q_2 = \frac{mv^2}{R} a \sin\theta - mga\cos\theta$$

$$\left(\simeq \frac{mv^2}{R} a - mga\alpha \quad \text{see below} \right)$$

[20%]

(b) Second gyro equation

$$A\underbrace{\cancel{\dot{\Omega}_2}}_{\substack{0 \\ \text{steady state}}} + (\underbrace{\cancel{A\Omega_3}}_{\substack{\text{fast} \\ \text{spin}}} - C\omega_3)\Omega_1 = Q_2$$

and with $\Omega_1 = -\dot{\phi}\sin\theta$

$$\therefore \quad C\omega_3 \dot{\phi} \sin\theta = \frac{mv^2}{R} a \sin\theta - mga\cos\theta$$

put $\theta = \frac{\pi}{2} - \alpha$, α small

$$\therefore \quad \sin\theta = \cos\alpha \approx 1$$

$$\text{and} \quad \cos\theta = \sin\alpha \approx \alpha$$

$$\therefore \quad C\omega_3 \dot{\phi} \simeq \frac{mv^2}{R} a - mga\alpha$$

3. cont No slip condition at P

$$\underline{v}_P = 0 = \underline{v}_G + \underset{\sim}{\omega} \times \underline{r}_{GP}$$

$$= V\underline{j} + (\omega_1 \underline{i} + \omega_2 \underline{j} + \omega_3 \underline{k}) \times a\underline{i}$$

$$= V\underline{j} - \omega_2 a \underline{k} + \omega_3 a \underline{j}$$

$$\therefore \quad \omega_2 = 0$$

$$\omega_3 = -\frac{V}{a}$$

and $\dot{\phi} = \frac{V}{R}$

$$\therefore \quad C\left(-\frac{V}{a}\right)\frac{V}{R} \simeq \frac{mV^2}{R}a - mga\alpha$$

$$\therefore \quad (C + ma^2)\frac{V^2}{R} \simeq mga^2\alpha$$

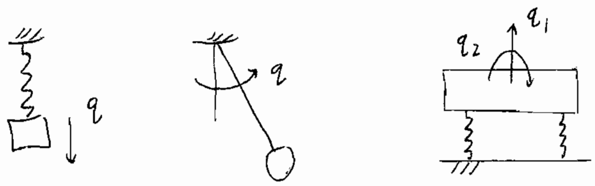
$$\therefore \quad \alpha \simeq \frac{(C+ma^2)V^2}{mga^2R}$$

and for $C = ma^2$

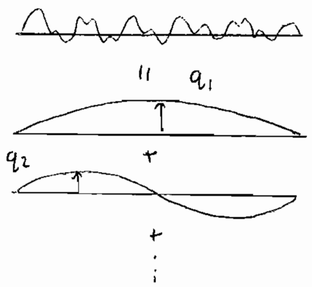
$$\alpha \simeq \frac{2V^2}{gR}$$ [80%]

4 a) The displacements of a dynamic system are described by a set of <u>generalised coordinates</u>. Examples are:-

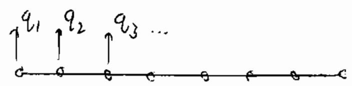
Simple systems



Complex systems



or



(as used in the finite Element method)

[15%]

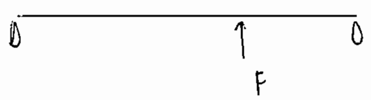
b) Generalised Force Q_i is the force appearing on the right hand side of the equation of motion for q_i, arising from external forces. Can be found from:-

$$\sum_i Q_i \, \delta q_i = \delta W \cdot$$ work done by the external forces when the system is moved through small virtual displacements δq_i

[15%]

c)

$$u(x_0,t) = q_1 \sin \pi x_0/L + q_2 \sin 2\pi x_0/L$$



Work done $= \delta u . F = F\left[\delta q_1 . \sin \pi x_0/L + \delta q_2 \sin 2\pi x_0/L\right] = Q_1 \delta q_1 + Q_2 \delta q_2$

$\Rightarrow$ $\underline{Q_1 = F \sin \pi x_0/L}$; $\underline{Q_2 = F \sin 2\pi x_0/L}$ [15%]

d)

$$\dot{u} = \dot{q}_1 \sin \pi x_0/L + \dot{q}_2 \sin 2\pi x_0/L$$

Additional kinetic energy $= \frac{1}{2} M \dot{u}^2 = \frac{1}{2} M \dot{q}_1^2 \sin^2 \pi x_0/L + \frac{1}{2} M \dot{q}_2^2 \sin^2 2\pi x_0/L + M \dot{q}_1 \dot{q}_2 \sin \pi x_0/L . \sin 2\pi x_0/L$

$= T_{add}$

$$\Rightarrow \quad T = \left(\frac{ML}{4}\right)(\dot{q}_1^2 + \dot{q}_2^2) + T_{add} \qquad ; \quad V = \frac{\pi^2 EI}{4L}\{q_1^2 + q_2^2 4\}$$

Equations of motion
$$\underbrace{\begin{pmatrix} \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j} \end{pmatrix}}_{\text{mass matrix}} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{\partial^2 U}{\partial q_i \partial q_j} \end{pmatrix}}_{\text{stiffness matrix}} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{1}{2}ML + M\sin^2 \pi x_0/L & M\sin\pi x_0/L \, . \, \sin 2\pi x_0/L \\ M\sin\pi x_0/L \, . \, \sin 2\pi x_0/L & \frac{1}{2}ML + M\sin^2 2\pi x_0/L \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\pi^2 EI/L & 0 \\ 0 & 2\pi^2 EI/L \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$$

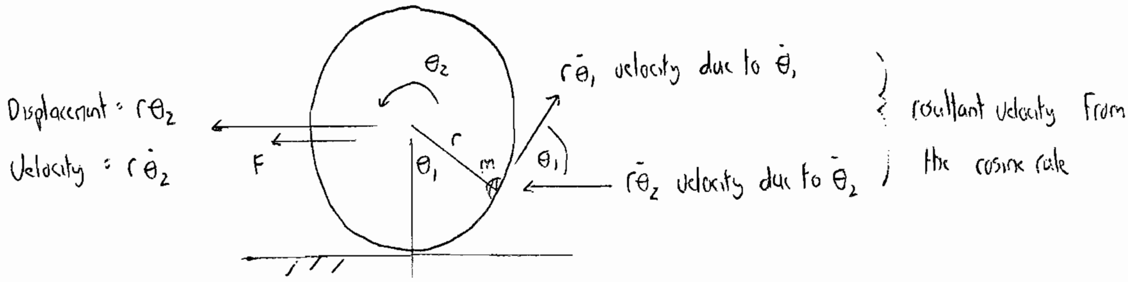
For $x_0 = L/2 \Rightarrow$
$$\begin{pmatrix} \frac{1}{2}ML + M & 0 \\ 0 & \frac{1}{2}ML \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\pi^2 EI/L & 0 \\ 0 & 2\pi^2 EI/L \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$$

$$\Rightarrow \quad \underline{\omega_1^2 = \frac{\frac{1}{2}\pi^2 EI/L}{\frac{1}{2}ML + M}} \qquad \underline{\omega_2^2 = \frac{4\pi^2 EI}{ML^2}}$$ [30%]

e) $\quad T_{ad} = \frac{1}{2}M\dot{u}^2 = \frac{1}{2}M\left\{\sum \dot{q}_j \sin(j\pi x/L)\right\}^2$

$$\underline{M_{ij} = \frac{\partial^2 T_{add}}{\partial \dot{q}_i \partial \dot{q}_j} = M\sin(i\pi x/L)\sin(j\pi x/L)}$$ [25%]

5. a)



Resultant vel^2 of m $= (r\dot\theta_1)^2 + (r\dot\theta_2)^2 - 2(r\dot\theta_1)(r\dot\theta_2)\cos\theta_1$

$$\Rightarrow \quad T = \underbrace{\tfrac{1}{2}mr^2\left[\dot\theta_1^2 + \dot\theta_2^2 - 2\dot\theta_1\dot\theta_2\cos\theta_1\right]}_{\text{For mass m}} + \underbrace{\tfrac{1}{2}M(r\dot\theta_2)^2}_{\text{Translational motion of cylinder}} + \underbrace{\tfrac{1}{2}(Mr^2)\dot\theta_2^2}_{\text{Rotational motion of cylinder}}$$

$$\underline{T = \tfrac{1}{2}mr^2\left[\dot\theta_1^2 + \dot\theta_2^2 - 2\dot\theta_1\dot\theta_2\cos\theta_1\right] + Mr^2\dot\theta_2^2}$$ [15%]

b) Potential energy $\underline{V = -mgr\cos\theta_1}$

For θ_1: $$\frac{d}{dt}\left[\frac{\partial T}{\partial\dot\theta_1}\right] - \frac{\partial T}{\partial\theta_1} + \frac{\partial V}{\partial\theta_1} = 0$$

$$\frac{d}{dt}\left[mr^2\dot\theta_1 - mr^2\dot\theta_2\cos\theta_1\right] - mr^2\dot\theta_1\dot\theta_2\sin\theta_1 + mgr\sin\theta_1 = 0$$

$$\Rightarrow \quad \underline{mr^2\ddot\theta_1 - mr^2\ddot\theta_2\cos\theta_1 + mgr\sin\theta_1 = 0} \quad \text{——} \quad (1)$$

For θ_2: $$\frac{d}{dt}\left[\frac{\partial T}{\partial\dot\theta_2}\right] - \frac{\partial T}{\partial\theta_2} + \frac{\partial V}{\partial\theta_2} = Q_2$$

$$\frac{d}{dt}\left[mr^2\dot\theta_2 - mr^2\dot\theta_1\cos\theta_1 + 2Mr^2\dot\theta_2\right] = Fr \qquad \left[\delta W = Q_2\delta\theta_2 = Fr\delta\theta_2 \Rightarrow Q_2 = Fr\right]$$

$$\Rightarrow \quad \underline{(mr^2 + 2Mr^2)\ddot\theta_2 - mr^2\ddot\theta_1\cos\theta_1 + mr^2\dot\theta_1^2\sin\theta_1 = Fr} \quad \text{——} \quad (2)$$ [20%]

c) For $\dot\theta_1 = \ddot\theta_2 = 0$, (1) $\Rightarrow$ $-mr^2\ddot\theta_2\cos\theta_1 + mgr\sin\theta_1 = 0 \Rightarrow \ddot\theta_2 = (g/r)\tan\theta_1$
(2) $\Rightarrow$ $(m+2M)r^2\ddot\theta_2 = Fr$
$\Big\}\; F = (m+2M)g\tan\theta_1$

$\Rightarrow$ For $\theta_1 = \psi$, $\underline{F = (m+2M)g\tan\psi}$ [30%]

d) For small amplitude vibrations:

$$\begin{array}{l} (1) \rightarrow \\ (2) \rightarrow \end{array} \quad \begin{pmatrix} Mr^2 & -Mr^2 \\ -Mr^2 & (M+2M)r^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + \begin{pmatrix} Mgr & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left| -\omega^2 [M] + [k] \right| = 0 \Rightarrow \begin{vmatrix} -\omega^2 Mr^2 + Mgr & -\omega^2 Mr^2 \\ -\omega^2 Mr^2 & -\omega^2 (M+2M) r^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad \omega^2 \left[-(M+2M) r^2 \left(-\omega^2 Mr^2 + Mgr \right) - \omega^2 M^2 r^4 \right] = 0$$

$$\omega^2 \left[-\omega^2 r^4 M (\;..\; 2M) + (M+2M) Mgr^3 \right] = 0$$

$$\Rightarrow \quad \underline{\omega = 0} \quad \text{or} \quad \underline{\omega = \left(\frac{g}{r} \right) \left(\frac{M+2M}{\;....\; 2M} \right)}$$

[35%]

3C5 Dynamics: Answers to Tripos Paper 2009

1. (a) AAA body with A$=28ma^2/5$.

   (b) $I_{zz}=16ma^2/3, \quad I_{xy}=0, \quad I_{yz}=-4ma^2$.

2. (a) $\dot{\phi}=mga/(C\omega)$.

   (b) (i) $\dot{\mathbf{h}}=Q\mathbf{K}$, (ii) $\dot{\theta}=-Q/(C\omega)$.

3. (a) $Q=(mV^2a/R)\cos\alpha-mga\sin\alpha$.

4. (c) $Q_1=F\sin(\pi x_0/L), \quad Q_2=F\sin(2\pi x_0/L)$.

   (d) Additional $T=0.5[\dot{q}_1\sin(\pi x_0/L)+\dot{q}_2\sin(2\pi x_0/L)]^2$,

   $\omega_1^2=\pi^2 EI/(mL^2+2ML), \quad \omega_2^2=4\pi^2 EI/(mL^2)$.

   (e) $M_{jk}=M\sin(j\pi x_0/L)\sin(k\pi x_0/L)$.

5. (b) $mr^2\ddot{\theta}_1-mr^2\ddot{\theta}_2\cos\theta_1+mgr\sin\theta_1=0,$

   $(mr^2+2Mr^2)\ddot{\theta}_2-mr^2\ddot{\theta}_1\cos\theta_1+mr^2\dot{\theta}_1^2\sin\theta_1=Fr$.