2009, PART IIA, PAPER 3C6 VIBRATION, (D. CEBON)

1 (a) From Data Sheet: $\rho A \dfrac{\partial^2 w}{\partial t^2} + EI \dfrac{\partial^4 w}{\partial x^4} = 0$

Try $w = u(x)e^{i\omega t}$, and set $\alpha^4 = \dfrac{\rho A \omega^2}{EI}$ (1)

Then $u'''' = \alpha^4 u$, with general solution

$u = K_1 \cos\alpha x + K_2 \sin\alpha x + K_3 \cosh\alpha x + K_4 \sinh\alpha x.$

At $x=0$: $\begin{cases} u=0 \rightarrow K_1 + K_3 = 0 \\ u'=0 \rightarrow K_2 + K_4 = 0 \end{cases}$

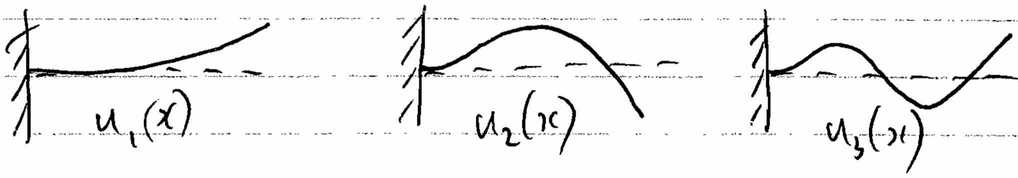
At $x=L$: $\begin{cases} u''=0 \rightarrow K_1(-\cos\alpha L - \cosh\alpha L) + K_2(-\sin\alpha L - \sinh\alpha L) = 0 \\ u'''=0 \rightarrow K_1(\sin\alpha L - \sinh\alpha L) + K_2(-\cos\alpha L - \cosh\alpha L) = 0 \end{cases}$

Determinant $=0$, so $(\cos\alpha L + \cosh\alpha L)^2 = -(\sin\alpha L - \sinh\alpha L) \times (\sin\alpha L + \sinh\alpha L)$

which simplifies to $\cos\alpha L \cosh\alpha L = -1$

Roots given α, which give ω from (1).

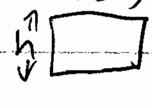
Sketches:



(b) From Data sheet:

$V = \frac{1}{2}E\int I w''^2 dx$, $T = \frac{1}{2}\rho\int A \dot{w}^2 dx$

where $I = \frac{1}{12}h^2 \times A$ with $A = bh$



So Rayleigh quotient is:

$$\omega^2 \simeq \frac{\frac{Eb}{24}\int h^3 w''^2 dx}{\frac{1}{2}\rho b\int h w^2 dx} = \frac{E}{12\rho}\frac{\int h^3 w''^2 dx}{\int h w^2 dx}$$

For $h = h_0 + \delta h$, $|\delta h| \ll h_0$, can use original mode $u_1(x)$ from (a) as an approximation to the new mode.

1 (cont)

So
$$\omega^2 \simeq \frac{E}{12\rho} \frac{\int (h_0^3 + 3h_0^2\,\delta h)\, u_1''^2\, dx}{\int (h_0 + \delta h)\, u_1^2\, dx}$$

$$= \frac{V_0 + \delta V}{T_0 + \delta T} \quad \text{say}$$

$$\simeq (V_0 + \delta V)\, \frac{1}{T_0}\left(1 + \frac{\delta T}{T_0}\right)^{-1} \simeq (V_0 + \delta V)\, \frac{1}{T_0}\left(1 - \frac{\delta T}{T_0}\right) \text{ (binomial)}$$

$$\simeq \frac{V_0}{T_0} + \frac{\delta V}{T_0} - \frac{V_0\,\delta T}{T_0^2} = \frac{V_0}{T_0} + \left(\delta V - \frac{V_0}{T_0}\delta T\right)\frac{1}{T_0}$$

But $\frac{V_0}{T_0} = \omega_1^2$, the original frequency.

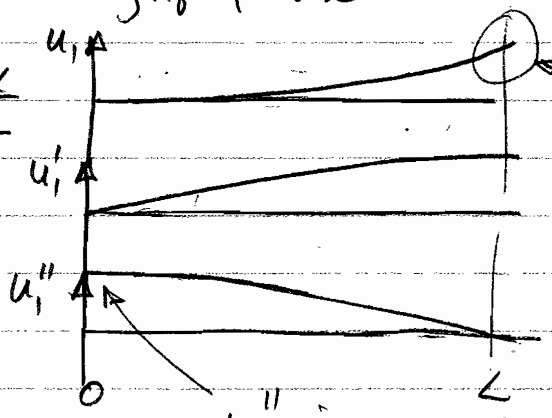
So
$$\omega^2 \simeq \omega_1^2 + \frac{1}{T_0}\left(\delta V - \omega_1^2\, \delta T\right)$$

$$= \omega_1^2 + \frac{E}{12\rho} \frac{\left[\int 3h_0^2\,\delta h\, u_1''^2\, dx - \omega_1^2 \int \delta h\, u_1^2\, dx\right]}{\int h_0\, u_1^2\, dx} \qquad (1)$$

(c) $u_1(x) \approx 1 - \cos\frac{\pi x}{2L}$

$\Rightarrow u_1' \approx \frac{\pi}{2L} \sin\frac{\pi x}{2L}$

$\Rightarrow u_1'' \approx \frac{\pi^2}{4L^2} \cos\frac{\pi x}{2L}$



u_1 is max at tip

u_1'' is max at root

From (1):

+ve δh at root gives max increase in u_1'' $\Rightarrow$ $\omega \uparrow$

Removed $\rightarrow$ −ve δh at root gives max decrease in u_1'' $\Rightarrow$ $\omega \downarrow$

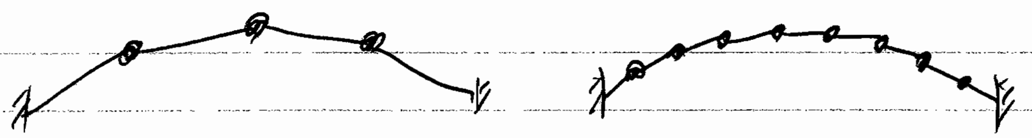
+ve δh at tip gives max increase in u_1 $\Rightarrow$ $\omega \downarrow$

Removal $\rightarrow$ −ve δh at tip gives max decrease in u_1 $\Rightarrow$ $\omega \uparrow$

2 (a) (Bookwork)

Discrete	Continuous
Finite no. of DoF (generalised coordinates)	Continuous displacement e.g. $w(x)$
Matrix equations "coupled oscillators"	Partial differential equations
Finite number of modes = no. of DoF	Infinite number of modes
Can prove <u>general</u> results	Each problem looks different, so do specific examples

Any continuous system can be approached as a limit of some kind of a discrete system with more + more DoF. E.g. string:



etc

Finite difference or finite element models are doing this explicitly, solving discrete approximations to continuous systems.

(b) From data sheet: $\rho A \frac{\partial^2 w}{\partial t^2} - EA \frac{\partial^2 w}{\partial x^2} = 0$

For modes, $w = u(x) e^{i\omega t}$

$\therefore \quad c^2 u'' = -\omega^2 u \quad$ where $c^2 = E/\rho$

General solution $u = K_1 \cos \frac{\omega x}{c} + K_2 \sin \frac{\omega x}{c}$

2 (cont)

At $x=0$, $u=0$ $\therefore K_1 = 0$

At $x=L$, $u'=0$ $\therefore \frac{K_2}{c}\cos\frac{\omega L}{c} = 0$ (zero stress)

$$\therefore \frac{\omega L}{c} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \dots = (n-\tfrac{1}{2})\pi \quad n=1,2,3\dots$$

Mode shapes $u_n(x) = K_2 \sin\frac{\omega_n x}{c} = K_2 \sin\frac{(n-\frac{1}{2})\pi x}{L}$

(c) From data sheet, impulse response at y due to forcing at x is $g = \sum \frac{u_n(x)\,u_n(y)}{\omega_n} \sin\omega_n t$

Need to normalise modes so $\int u_n^2(x)\,dm = 1$

ie $$\int_0^L K_2^2 \sin^2\left(\frac{(n-\frac{1}{2})\pi x}{L}\right) \rho A\,dx = 1$$

$$\therefore K_2^2 = \frac{1}{\rho A \int_0^L \sin^2\frac{(n-\frac{1}{2})\pi x}{L}\,dx} = \frac{1}{\rho A \cdot L/2}$$

So normalised modes are $u_n(x) = \sqrt{\frac{2}{\rho A L}} \sin\frac{(n-\frac{1}{2})\pi x}{L}$

So response $$g(t) = \frac{2}{\rho A L}\sum_n \sin(n-\tfrac{1}{2})\pi \cdot \sin\frac{(n-\frac{1}{2})\pi}{4} \frac{\sin\omega_n t}{\omega_n}$$

since $x = L$ and $y = L/4$

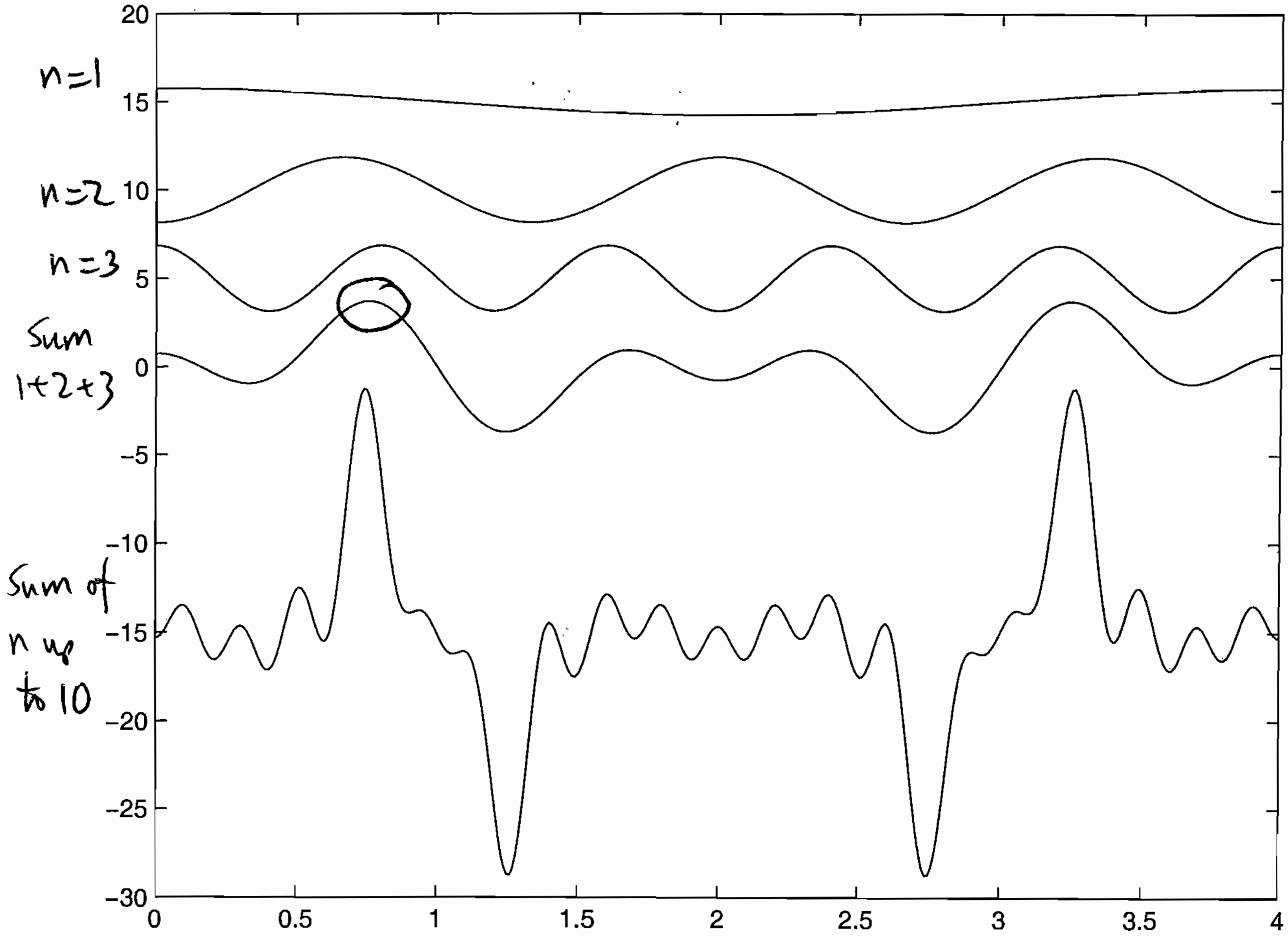
So velocity $$\frac{dg}{dt} = \frac{2}{\rho A L}\sum_n (-1)^n \sin\frac{(n-\frac{1}{2})\pi}{4}\cos\omega_n t$$

$\sin\frac{(1-\frac{1}{2})\pi}{4} = 0.383$, $\sin\frac{(2-\frac{1}{2})\pi}{4} = 0.924$, $\sin\frac{(3-\frac{1}{2})\pi}{4} = 0.924$

2 (cont)

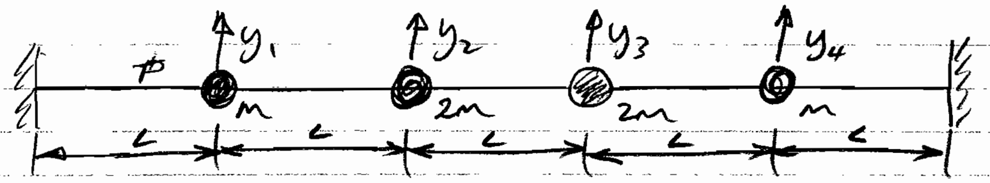
Plot of first 3 terms, sum of those, and sum up to 10 modes.

Numbers correspond to $L=1$, $\rho A=1$, $c=1$

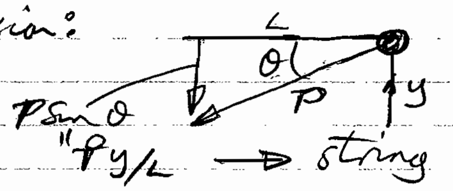


Peak occurs at about 0·7 on this scale : shown as ring.
This time delay is simply the time for a wave at speed c to travel from $x=L$ to $x=L/4$, so a distance $\frac{3L}{4}$, and a time $\frac{3L}{4c}$ $=0{\cdot}75$ on this plot.

3.

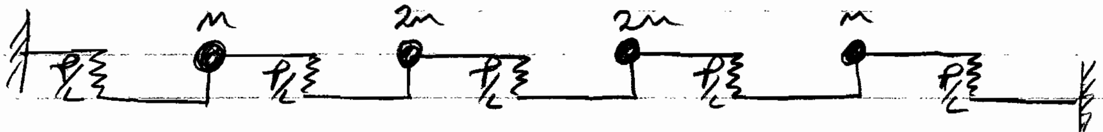


(a) Effect of tension:



$\sin\theta = y/L$

$\rightarrow$ string acts as a stiffness of $\frac{Py}{L}$



$$\underline{PE}: \; V = \tfrac{1}{2} P/L \left[y_1^2 + (y_2-y_1)^2 + (y_3-y_2)^2 + (y_4-y_3)^2 + y_4^2 \right]$$

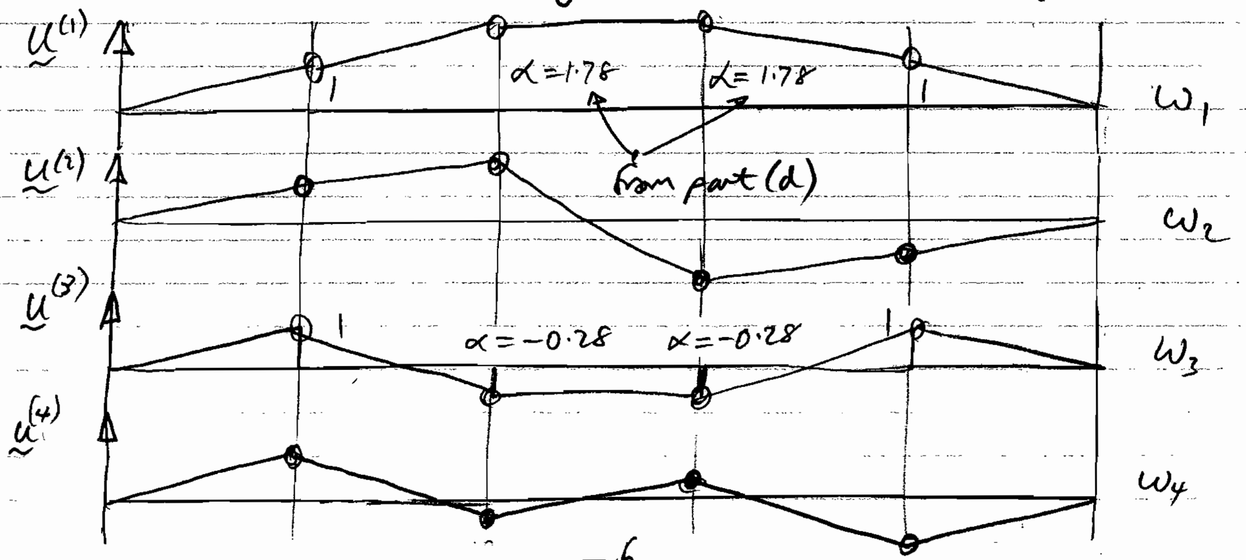
$$= \tfrac{1}{2} P/L \left[2y_1^2 + 2y_2^2 + 2y_3^2 + 2y_4^2 - 2y_1y_2 - 2y_2y_3 - 2y_3y_4 \right]$$

$$= \tfrac{1}{2} \left[y_1 \; y_2 \; y_3 \; y_4 \right] \underbrace{P/L \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}}_{[K]} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

$$\underline{KE} \; T = \tfrac{1}{2} m \left(\dot{y}_1^2 + 2\dot{y}_2^2 + 2\dot{y}_3^2 + \dot{y}_4^2 \right)$$

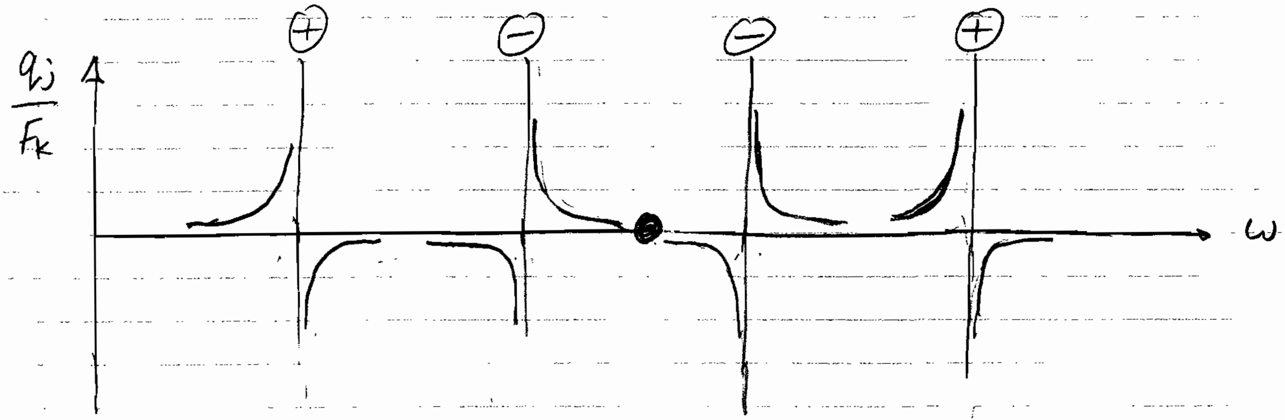
$$= \tfrac{1}{2} \left[\dot{y}_1 \; \dot{y}_2 \; \dot{y}_3 \; \dot{y}_4 \right] \underbrace{m \begin{bmatrix} 1 & & & 0 \\ & 2 & & \\ & & 2 & \\ 0 & & & 1 \end{bmatrix}}_{[M]} \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{Bmatrix}$$

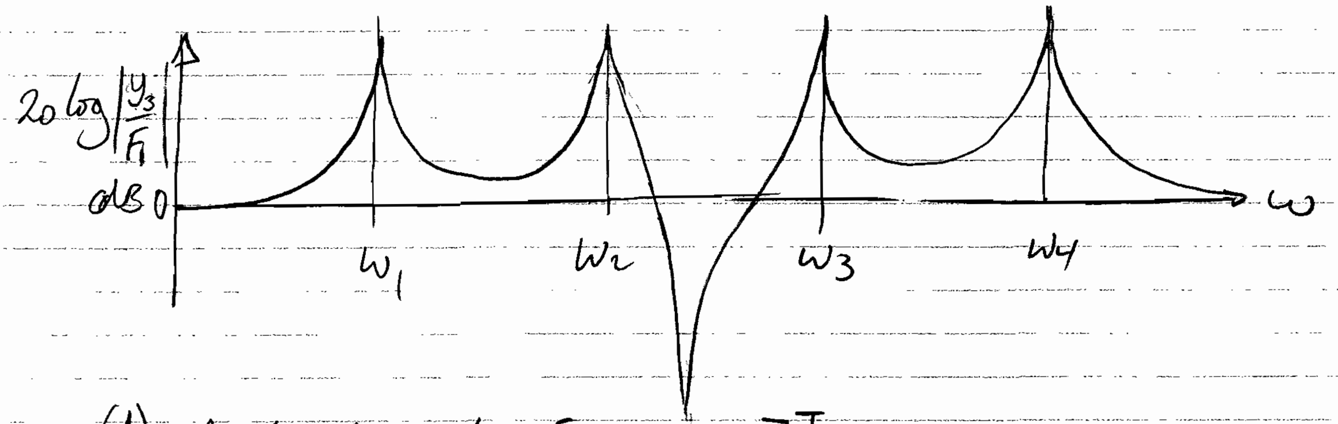
(b) All modes are symmetric or anti-symmetric



3(cont) (c) $q_j/F_k = \sum_N \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2}$ with $j=3$, $k=1$

Signs of $u_j^{(n)} u_k^{(n)}$ are: $[+, -, -, +]$





(d) Rayleigh with $[1\ \alpha\ \alpha\ 1]^T$

$$R = \frac{V}{T^*} = \frac{\frac{1}{2}\frac{P}{L}\left[1 + (\alpha-1)^2 + \cancel{(\alpha-\alpha)^2} + (\alpha-1)^2 + 1\right]}{\frac{1}{2}m\left[1 + 2\alpha^2 + 2\alpha^2 + 1\right]}$$

$$= \frac{P}{Lm}\left(\frac{2\alpha^2 - 4\alpha + 4}{4\alpha^2 + 2}\right) = \frac{P}{Lm}\left(\frac{\alpha^2 - 2\alpha + 2}{2\alpha^2 + 1}\right)$$

Minimise R : $\frac{dR}{d\alpha} = 0 = \frac{(2\alpha^2+1)(2\alpha-2) - (\alpha^2 - 2\alpha + 2)4\alpha}{(2\alpha^2+1)^2}$

ie $2\alpha^2 - 3\alpha - 1 = 0 \Rightarrow \alpha = \frac{3 \pm \sqrt{17}}{4} = 1.78, -0.28$

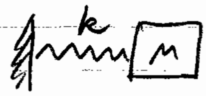
$$\therefore \omega_1^2 = \frac{P}{Lm}\left(\frac{1.78^2 - 2(1.78) + 2}{2(1.78)^2 + 1}\right) = \underline{\underline{0.219 \frac{P}{Lm}}}$$

$$\& \ \omega_3^2 = \frac{P}{Lm}\left(\frac{(-0.28)^2 - 2(-0.28) + 2}{2(-0.28)^2 + 1}\right) = \underline{\underline{2.28 \frac{P}{Lm}}}$$

Both of these frequencies are exact since both mode shapes are symmetric and of the form $[1\ \alpha\ \alpha\ 1]^T$

4 (a) <u>Mode 1:</u> Rigid body translation $\omega_1 = 0$ $u^{(1)} = [1 \ 1 \ 1]^T$

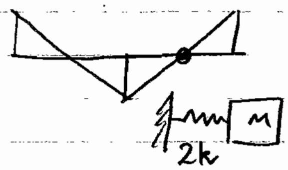
<u>Mode 2:</u> Anti-symmetric, mass 2 stationary (node)



$\omega_2 = \sqrt{k/m}$, $u^{(2)} = [1 \ 0 \ -1]^T$

<u>Mode 3:</u> Symmetric with 2 nodes. For zero net momentum, mode shape is: $[a \ -b \ a]^T$

$$ma + ma - 2mb = 0 \Rightarrow b = a \Rightarrow u^{(3)} = [1 \ -1 \ 1]^T$$

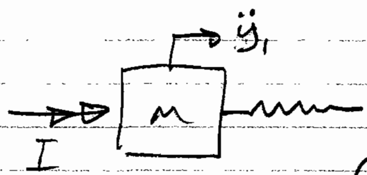


Node in middle of each spring

$$\Rightarrow \omega_3 = \sqrt{\frac{2k}{m}}$$

(b) When system is excited in mode 2, the response at the other end must be the same magnitude but 180° out of phase ie $-\Delta$. (NB no damping is needed to maintain a finite response for a displacement input).

(d) Consider mass 1 during the impact



$$m\ddot{y}_1 + k(y_1 - y_2) = I\delta(t)$$

$$\int_{0^-}^{0^+} m\ddot{y}_1\,dt + \underbrace{\int_{0^-}^{0^+} k(y_1 - y_2)}_{=0} = \int_{0^-}^{0^+} I\delta(t)\,dt$$

$$\text{ie } m\dot{y}_1(0^+) = I \Rightarrow \dot{y}_1(0^+) = I/m$$

So initial conditions are $y_1(0) = y_2(0) = y_3(0) = 0$

$$\dot{y}_1(0) = I/m, \quad \dot{y}_2(0) = \dot{y}_3(0) = 0$$

The subsequent motion will be a combination of the 3 modal responses. Motion after the impulse will be:

$$\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}}_{\text{e-vectors}} \begin{Bmatrix} at \\ b\sin\omega_2 t \\ c\sin\omega_3 t \end{Bmatrix} \qquad \text{——} (1)$$

4 (cont) Initial vels by differentiating

$$\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{Bmatrix} = \begin{Bmatrix} I/m \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{Bmatrix} a \\ \omega_2 b \\ \omega_3 c \end{Bmatrix} \qquad \text{—— } (2)$$

$$\Rightarrow \quad \text{Invert} \Rightarrow \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \frac{I}{m} \begin{Bmatrix} 1/4 \\ 1/2\omega_2 \\ 1/4\omega_3 \end{Bmatrix} \qquad \text{—— } (3)$$

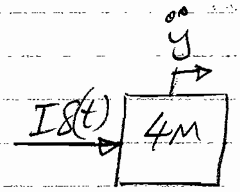
Substitute (3) into (2):

$$y_2(t) = \frac{I}{m}\left[\frac{t}{4} - \frac{1}{4\omega_3}\sin\omega_3 t\right]$$

At time $t = \sqrt{M/k}$, with $\omega_3 = \sqrt{2k/m}$

$$\Rightarrow y_2(t) = \frac{I}{4m}\left[\sqrt{\frac{M}{k}} - \sqrt{\frac{M}{2k}}\sin\sqrt{2}\right]$$

$$= \underline{\underline{0.075\frac{I}{\sqrt{Mk}}}}$$

(c) 

$$4m\ddot{y} = I\,\delta(t)$$

$$\int_{0^-}^{0^+} 4m\ddot{y}\,dt = I\int_{0^-}^{0^+} \delta(t)\,dt$$

$$4m\,\dot{y}(0^+) = I \qquad [\text{ie Impulse} = \Delta mv]$$

$$\Rightarrow \dot{y}(0^+) = I/4m \quad (\text{Const velocity})$$

ENGINEERING TRIPOS PART IIA

Module 3C6 Examination, 2009

Answers

1 (a) $\cos\alpha L \cosh\alpha L = -1$, (b) $\omega^2 = \omega_1^2 + \frac{E}{12\rho}\left[\frac{\int_0^L 3h_0^2\,\delta h\left(u_1''\right)^2 dx - \omega_1^2 \int_0^L \delta h\, u_1^2\, dx}{\int_0^L h_0\, u_1^2\, dx}\right]$.

2 (b) $\omega_n = \frac{1}{L}\sqrt{\frac{E}{\rho}}\left(n - \frac{1}{2}\right)\pi, \quad u_n(x) = K\sin\frac{(n-1/2)\pi x}{L}; \quad n = 1, 2, 3...$

(c) $\left.\frac{dg}{dt}\right|_{x=L/4} = \frac{2}{\rho A L}\sum_{n=1,2..}(-1)^n \sin\frac{(n-1/2)\pi}{4}\cos\omega_n t$; (d) $\frac{3L}{4}\sqrt{\frac{\rho}{E}}$.

3 (a) $V = \frac{P}{2L}\left[y_1^2 + (y_2 - y_1)^2 + (y_3 - y_2)^2 + (y_4 - y_3)^2 + y_4^2\right]$

$T = \frac{m}{2}\left[\dot{y}_1^2 + 2\dot{y}_2^2 + 2\dot{y}_3^2 + \dot{y}_4^2\right]$

(d) $\alpha = 1.78, -0.28; \quad \omega_1^2 = 0.219\frac{P}{Lm}$.

4 (a) $\omega_1 = 0, \quad u^{(1)} = [1 \quad 1 \quad 1]^T; \quad \omega_2 = \sqrt{\frac{k}{m}}, \quad u^{(1)} = [1 \quad 0 \quad -1]^T$;

$\omega_3 = \sqrt{\frac{2k}{m}}, \quad u^{(1)} = [1 \quad -1 \quad 1]^T$

(b) $-\Delta, -180°$ (c) $\dot{y}(0^+) = \frac{I}{4m}$ (d) $y_2\left(\sqrt{\frac{m}{k}}\right) = 0.075\frac{I}{\sqrt{mk}}$.