Prof N Fleck

ENGINEERING TRIPOS PART IIA

MODULE 3C7 : Mechanics of Solids

CRIB FOR EXAM 2008-9.

1. (b) The Lamé equations are:

$$\sigma_{rr} = A - \frac{B}{r^2} - \frac{3+\nu}{8}\rho\omega^2 r^2$$

$$\sigma_{\theta\theta} = A + \frac{B}{r^2} - \frac{1+3\nu}{8}\rho\omega^2 r^2$$

Now, $\sigma_{rr} = 0$ at $r = R$ and at $r = \alpha R$

$$\Rightarrow \quad A - \frac{B}{R^2} - \frac{3+\nu}{8}\rho\omega^2 R^2 = 0$$

$$A - \frac{B}{\alpha^2 R^2} - \frac{3+\nu}{8}\rho\omega^2 \alpha^2 R^2 = 0$$

subtract: $\left(\frac{1}{\alpha^2} - 1\right)\frac{B}{R^2} + \left(\frac{3+\nu}{8}\right)\rho\omega^2 R^2 \left(\alpha^2 - 1\right) = 0$

$$\Rightarrow \quad B = \left(\frac{3+\nu}{8}\right)\rho\omega^2 R^4 \alpha^2$$

$$A = \frac{B}{R^2} + \left(\frac{3+\nu}{8}\right)\rho\omega^2 R^2 = \left(\frac{3+\nu}{8}\right)\rho\omega^2 R^2 \left(1+\alpha^2\right)$$

check: $1+\alpha^2 - 1 - \alpha^2 = 0$ ✓

Hence, $\sigma_{rr} = \left(\frac{3+\nu}{8}\right)\rho\omega^2 R^2 \left(1 + \alpha^2 - \alpha^2\frac{R^2}{r^2} - \frac{r^2}{R^2}\right)$

and $\sigma_{\theta\theta} = \left(\frac{3+\nu}{8}\right)\rho\omega^2 R^2 \left(1 + \alpha^2 + \alpha^2\frac{R^2}{r^2} - \left(\frac{1+3\nu}{3+\nu}\right)\frac{r^2}{R^2}\right)$

(c) First yield occurs at $r = \alpha R$.
Then, $\sigma_{rr} = 0$ and

$$\sigma_{\theta\theta} = \sigma_Y = \frac{3+\nu}{8}\rho\omega^2 R^2 \left(\alpha^2 + 2 - \left(\frac{1+3\nu}{3+\nu}\right)\alpha^2\right)$$

Q1 (c) Write $\beta \equiv 2 + \alpha^2 - \left(\frac{1+3\nu}{3+\nu}\right)\alpha^2$

$$\Rightarrow \quad \omega_Y^2 = \left(\frac{8}{3+\nu}\right)\frac{\sigma_Y}{\beta\rho R^2}$$

(a) $m = \int_{\alpha R}^{R} [t\rho\, 2\pi r]\, dr = \pi\rho t R^2 (1-\alpha^2)$

$K = \int_{\alpha R}^{R} [\tfrac{1}{2}(t\rho\, 2\pi r)\, \omega^2 r^2]\, dr = \int_{\alpha R}^{R} \pi\rho t \omega^2 r^3\, dr$

$\Rightarrow K = \frac{\pi}{4}\rho t \omega^2 R^4 (1-\alpha^4)$

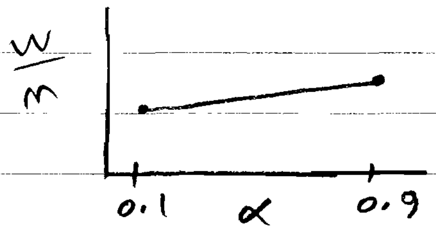
$\Rightarrow \frac{K}{m} = \frac{1}{4}\omega^2 R^2 (1+\alpha^2) = \left(\frac{2}{3+\nu}\right)\frac{\sigma_Y}{\rho}\frac{(1+\alpha^2)}{\beta}$

(d) What value of α maximises K/m?

Consider $Y \equiv \frac{1+\alpha^2}{\beta} = \frac{1+\alpha^2}{2+\alpha^2 - \left(\frac{1+3\nu}{3+\nu}\right)\alpha^2}$

$\alpha = 0$ gives $Y = \frac{1}{2}$

$= 1$ gives $Y = \frac{2}{3 - \left(\frac{1+3\nu}{3+\nu}\right)} = \frac{2}{3 - \frac{3}{5}} = \frac{10}{12} = \frac{5}{6}$



So $\alpha = 0.9$ is the optimal value.

(e) Choose a variable thickness $t(r)$ to get first yield everywhere.

2. (a) $\phi = r^{\lambda+1}\left[A\cos(\lambda+1)\theta + B\cos(\lambda-1)\theta\right]$

$$\nabla^2\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi$$

$$\Rightarrow r\frac{\partial\phi}{\partial r} = (\lambda+1)\, r^{\lambda+1}\left[A\cos(\lambda+1)\theta + B\cos(\lambda-1)\theta\right]$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = (\lambda+1)^2 r^{\lambda-1}\left[A\cos(\lambda+1)\theta + B\cos(\lambda-1)\theta\right]$$

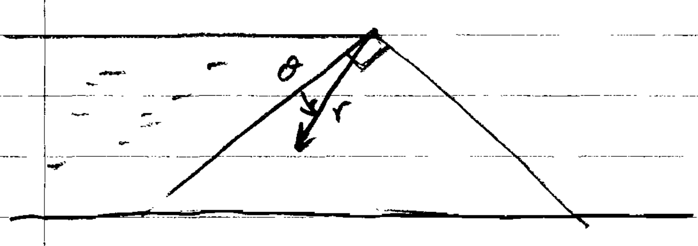
$$\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = -r^{\lambda-1}\left[(\lambda+1)^2 A\cos(\lambda+1)\theta + (\lambda-1)^2 B\cos(\lambda-1)\theta\right]$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} \qquad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

$$\frac{\partial^2\phi}{\partial r^2} = (\lambda+1)\lambda\, r^{\lambda-1}\left[A\cos(\lambda+1)\theta + B\cos(\lambda-1)\theta\right]$$

$$\sigma_{\theta\theta} = -\rho g\frac{r}{\sqrt{2}} = \frac{\partial^2\phi}{\partial r^2} \Rightarrow \lambda = 2$$

$$\frac{\partial\phi}{\partial\theta} = -r^{\lambda+1}\left[(\lambda+1)A\sin(\lambda+1)\theta + (\lambda-1)B\sin(\lambda-1)\theta\right]$$



So $6r(A+B) = -\rho g\frac{r}{\sqrt{2}} \Rightarrow A+B = \frac{-\rho g}{6\sqrt{2}}$

$\sigma_{r\theta} = 0$ on $\theta = 0, \pi/2$

$\left.\frac{\partial\phi}{\partial\theta}\right|_{\theta=0} = 0$ ✓

$$\left.\frac{\partial\phi}{\partial\theta}\right|_{\theta=\pi/2} = -r^3\left[3A\sin\frac{3\pi}{2} + B\sin\frac{\pi}{2}\right]$$

2. $\sigma_{r\theta}\Big|_{\theta=\pi/2} = 0 \Rightarrow \underline{B - 3A = 0} \Rightarrow B = 3A$

Also, $B + A = \frac{-\rho g}{6\sqrt{2}}$

$$\Rightarrow A = \frac{-\rho g}{24\sqrt{2}}, \quad B = \frac{-\rho g}{8\sqrt{2}}, \quad \lambda = 2$$

(b) $\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} = 6r\left[A\cos 3\theta + B\cos\theta\right]$

$$\Rightarrow \sigma_{\theta\theta} = \frac{-1}{4\sqrt{2}}\rho g r\left[\cos 3\theta + 3\cos\theta\right]$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\frac{1}{r}\left(\frac{\partial\phi}{\partial\theta}\right)$$

$$\frac{\partial\phi}{\partial\theta} = -r^3\left[3A\sin 3\theta + B\sin\theta\right]$$

$$\sigma_{r\theta} = 2r\left[3A\sin 3\theta + B\sin\theta\right]$$

$$\Rightarrow \sigma_{r\theta} = \frac{-\rho g r}{4\sqrt{2}}\left[\sin 3\theta + \sin\theta\right]$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\phi = r^3\left(A\cos 3\theta + B\cos\theta\right)$$

$$\frac{1}{r}\frac{\partial\phi}{\partial r} = 3r\left(A\cos 3\theta + B\cos\theta\right)$$

$$\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = -r\left(9A\cos 3\theta + B\cos\theta\right)$$

$$\Rightarrow \sigma_{rr} = r\left(-6A\cos 3\theta + 2B\cos\theta\right)$$

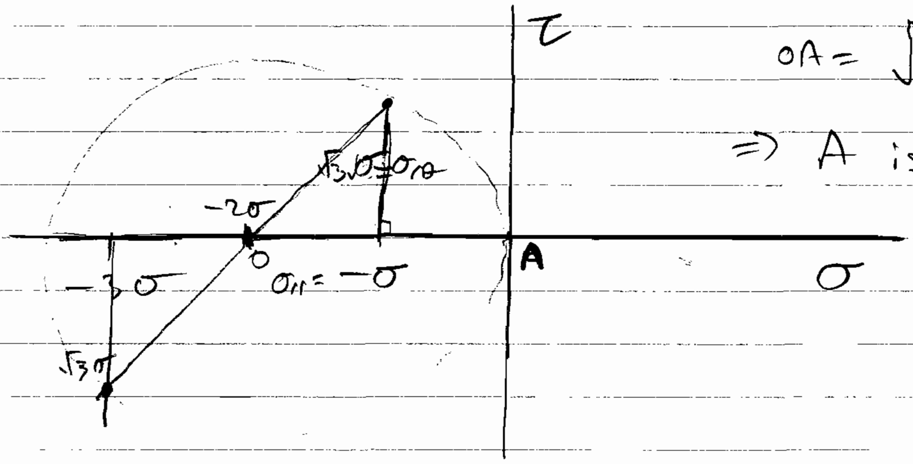
$$\sigma_{rr} = \frac{-\rho g r}{4\sqrt{2}}\left(-\cos 3\theta + \cos\theta\right)$$

2 (c) $\theta = 30°$

$$\Rightarrow \sigma_{rr} = -\frac{\rho g r}{4\sqrt{2}}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{8\sqrt{2}}\rho g r = -\sigma$$

$$\sigma_{\theta\theta} = -\frac{\rho g r}{4\sqrt{2}}\left(\frac{3\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{8\sqrt{2}}\rho g r = -3\sigma$$

$$\sigma_{r\theta} = -\frac{\rho g r}{4\sqrt{2}}\left(\frac{1}{2}+1\right) = -\frac{3}{8\sqrt{2}}\rho g r = -\sqrt{3}\sigma$$



$OA = \sqrt{\sigma^2 + (\sqrt{3}\sigma)^2} = 2\sigma$

$\Rightarrow$ A is at origin.

3. (a) $\phi = C\left(1-\frac{b^2}{r^2}\right)\left(r^2 - 2ar\cos\theta\right)$

$\nabla^2\phi = -2G\beta$

$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$

$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$

$\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = C\left(1-\frac{b^2}{r^2}\right)\left(\frac{2a}{r}\cos\theta\right)$

$\frac{\partial\phi}{\partial r} = C\,.\,\frac{2b^2}{r^3}\,.\left(r^2-2ar\cos\theta\right)$
$\quad + C\left(1-\frac{b^2}{r^2}\right)\left(2r - 2a\cos\theta\right)$

$\Rightarrow r\frac{\partial\phi}{\partial r} = C\,2b^2\left(1-\frac{2a}{r}\cos\theta\right)$
$\quad + C\left(1-\frac{b^2}{r^2}\right)\left(2r^2 - 2ar\cos\theta\right)$

$\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = C\,2b^2\,.\,\frac{2a}{r^2}\cos\theta + C\left(1-\frac{b^2}{r^2}\right)\left(4r-2a\cos\theta\right)$
$\quad + C\,\frac{2b^2}{r^3}\left(2r^2 - 2ar\cos\theta\right)$

$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = 4C\frac{b^2a}{r^3}\cos\theta + C\left(1-\frac{b^2}{r^2}\right)\left(4 - \frac{2a\cos\theta}{r}\right)$
$\quad + \frac{4Cb^2}{r^3}\left(r - a\cos\theta\right)$
$\quad = 4C - C\left(1-\frac{b^2}{r^2}\right)\frac{2a\cos\theta}{r}$

$\Rightarrow \nabla^2\phi = 4C = -2G\beta \Rightarrow C = -\frac{G\beta}{2}$

3 (b) $$\sigma_{zy} = -\frac{\partial \phi}{\partial x} = -\frac{\partial \phi}{\partial r} \quad \text{on} \quad \theta = 0$$

$$\phi = c \, . \left(1 - \frac{b^2}{r^2}\right)\left(r^2 - 2ar\cos\theta\right)$$

$$\Rightarrow \frac{\partial \phi}{\partial r} = c \, \frac{2b^2}{r^3}\left(r^2 - 2ar\cos\theta\right)$$
$$+ \, c\left(1 - \frac{b^2}{r^2}\right)\left(2r - 2a\cos\theta\right)$$

on $r = b, \theta = 0$ (inner concave face) we have

$$\underline{\sigma_{zy} = -2c\,(b - 2a)}$$

on $\theta = 0$, $r = 2a$ (outer convex face) we have

$$\sigma_{zy} = \cancel{-2c \, . \frac{b^2}{2a^3}\overbrace{\left(4a^2 - 4a^2\right)}^{=0}}$$
$$- \, c\left(1 - \frac{b^2}{4a^2}\right)\left(4a - 2a\right)$$

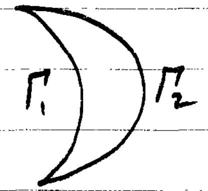
$$\Rightarrow \underline{\sigma_{zy} = -2c\left(1 - \frac{b^2}{4a^2}\right) . a}$$

$\phi =$ constant on boundary ?

$$(r\cos\theta - a)^2 + r^2\sin^2\theta = a^2 \quad \text{on } \Gamma_2$$

$$\Rightarrow \quad r^2 - 2ar\cos\theta = 0 \qquad r = 2a\cos\theta \quad \text{on } \Gamma_2$$

So $\phi = 0$ on Γ_1 $\qquad r = b$ on Γ_1



(b) $\quad dA = r\,d\theta\,dr$

$$T = 4\int_0^\alpha d\theta \int_b^{2a\cos\theta} dr\,(r\phi) = 4\int dr \int d\theta\,(r\phi)$$

where $\cos\alpha = a/b$

$\cos\alpha = a/b$

$$\int dr\, r\phi = C\int (r^2 - b^2)(r - 2a\cos\theta)\,dr$$

$$= C\int \left(r^3 - b^2 r - r^2\, 2a\cos\theta + 2ab^2\cos\theta\right) dr$$

$$= C\left[\tfrac{1}{4}r^4 - \tfrac{1}{2}b^2 r^2 - \tfrac{2}{3}r^3 a\cos\theta + 2ab^2 r\cos\theta\right]$$

4 (a) Postulate a collapse mechanism and thereby calculate ~~the~~ forming load in a complex operation. Upper bound ⇒ the process is achievable under the design load.

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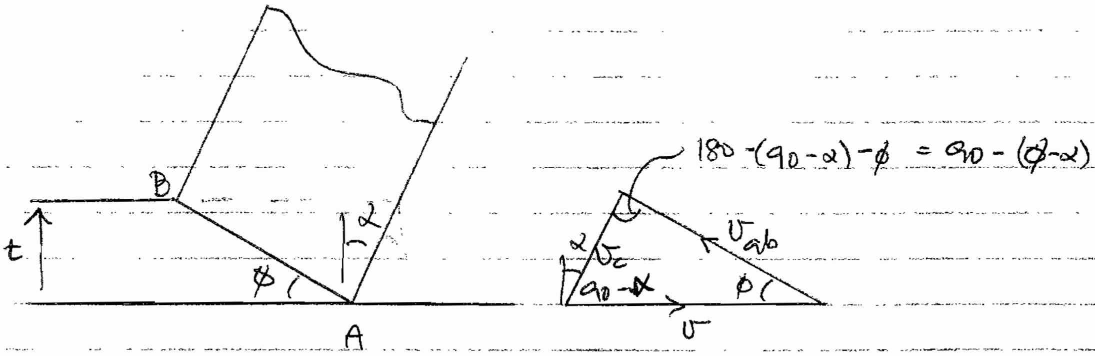
4.(b)



If no friction $\quad \dot{W} = k(l_{ab} \cdot v_{ab})$

But $\quad l_{ab} = \dfrac{t}{\sin\phi}$; $\dfrac{v_{ab}}{\sin(90-\alpha)} = \dfrac{v}{\sin[90-(\phi-\alpha)]}$

$$\therefore \quad \dot{W} = kvt \; \frac{1}{\sin\phi} \cdot \frac{\cos\alpha}{\sin[90-(\phi-\alpha)]}$$

$$\underbrace{\frac{\dot{W}}{kvt}} \Rightarrow \frac{\cos\alpha}{\sin\phi \cos(\phi-\alpha)}$$

will have minimum when

$$\frac{d}{d\phi} \sin\phi \cos(\phi-\alpha) \Rightarrow 0$$

i.e. $\quad -\sin\phi \sin(\phi-\alpha) + \cos\phi\cos(\phi-\alpha) = 0$

i.e. $\quad \tan\phi \tan(\phi-\alpha) = 1$

or $\quad \tan\phi = \cot(\phi-\alpha) \equiv \tan\left[\frac{\pi}{2}-(\phi-\alpha)\right]$

$$\phi = \frac{\pi}{2} - (\phi-\alpha)$$

$$\therefore \quad 2\phi = \frac{\pi}{2} + \alpha$$

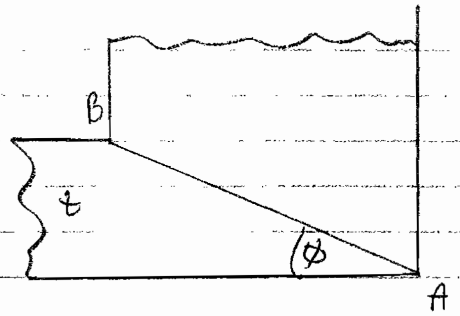
$$\underbrace{\phi = \frac{\pi}{4} + \frac{\alpha}{2}}$$ (Merchant Soln)

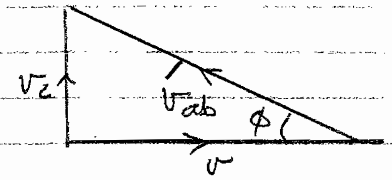
Now – in presence of friction

$$\dot{W} = k(l_{ab} \cdot v_{ab} + l \cdot v_c)$$

Now set $\alpha \Rightarrow 0$

4 (c)





$$l_{ab} = \frac{t}{\sin\phi} \qquad v_{ab} = \frac{v}{\cos\phi} \qquad v_z = v\tan\phi$$

$$\therefore \quad \dot{W} = k\left\{ \frac{t}{\sin\phi}\,\frac{v}{\cos\phi} + l\,v\tan\phi \right\}$$

$$\text{i.e.} \quad \frac{\dot{W}}{kvt} = \frac{1}{\sin\phi\cos\phi} + l/t \tan\phi$$

Let $\bar{l} \equiv l/t$ $\qquad (\dot{w}/kvt) = \dfrac{1 + \bar{l}\sin^2\phi}{\sin\phi\cos\phi}$

$$\text{i.e.} \quad (\dot{w}/kvt) = \frac{2 + 2\bar{l}\sin^2\phi}{2\sin\phi\cos\phi}$$

$$2\sin\phi\cos\phi \equiv \sin 2\phi \qquad 2\sin^2\phi = 1 - \cos 2\phi$$

$$\therefore \quad \dot{W}/kvt = \frac{2 + \bar{l}(1-\cos 2\phi)}{\sin 2\phi}$$

$$= \frac{(2+\bar{l}) - \bar{l}\cos 2\phi}{\sin 2\phi}$$

$$= (2+\bar{l})\csc 2\phi - \bar{l}\cot 2\phi$$

Put $\theta = 2\phi$ $\qquad (2+\bar{l})\csc\theta - \bar{l}\cot\theta$

$$\therefore \quad \frac{d(\;)}{d\theta} = -(2+\bar{l})\csc\theta\cot\theta + \bar{l}\csc^2\theta = 0$$

at min.

$$\therefore \quad (2+\bar{l})\cancel{\csc\theta}\cot\theta = \bar{l}\csc^2\theta$$

check if $l \to 0$
$\cos 2\phi = 0$
$\phi = \pi/4$ ✓ ok

$$\text{i.e.} \quad \cos\theta = \cos 2\phi = \frac{\bar{l}}{2+\bar{l}} = \frac{l}{2t+l}$$