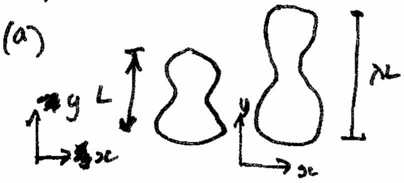
Mr F A McRobie

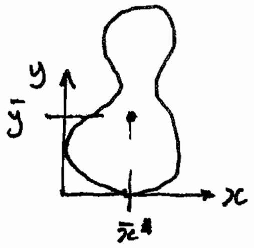
3D4 STRUCTURAL ANALYSIS AND STABILITY, 2009.

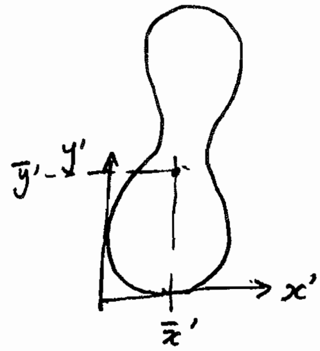
3D4 2009

Q1 (a)



Centroid





<u>Centroids</u>: $\bar{x} = \dfrac{\int x\,dA}{\int dA} = \dfrac{\int_A x\,dx\,dy}{\int_A dx\,dy}$ $\qquad \bar{x}' = \dfrac{\int_{A'} x'\,dx'\,dy'}{\int_{A'} dx'\,dy'}$

Now $\left.\begin{array}{l} x' = x \\ y' = \lambda y \end{array}\right\}$ $\Rightarrow$ $\bar{x}' = \dfrac{\int x\,dx\,\lambda\,dy}{\int dx.\lambda\,dy} = \dfrac{\lambda\int_A x\,dx\,dy}{\lambda\int_A dx\,dy}$

$= \dfrac{\int_A x\,dx\,dy}{\int_A dx\,dy} = \bar{x}$, (obviously).

$\bar{y}' = \dfrac{\int_{A'} y'\,dx'\,dy'}{\int_{A'} dx'\,dy'} = \int_A \dfrac{\lambda y\,dx.\lambda\,dy}{\int_A dx.\lambda\,dy} = \dfrac{\lambda^2}{\lambda}\dfrac{\int_A y\,dx\,dy}{\int_A dx\,dy}$

$= \lambda\,\bar{y}$ (again, fairly obvious).

2^{nd} Moments of area.

Either: Move to new coords centred at centroid

or stay with orig coords and use parallel axis theorem.

(Use latter).

$I_{x'x'} = \int y'^2\,dx'\,dy' = \lambda^3 I_{xx}$

$I_{y'y'} = \int x'^2\,dx'\,dy' = \lambda\ I_{yy}$

$I_{x'y'} = \int x'y'\,dx'\,dy' = \lambda^2\ I_{xy}$.

Q1 (a) con't'd

$$I_{x'x'} = I_{x'x'} - A'(\bar{y}')^2$$

about axis thru centroid

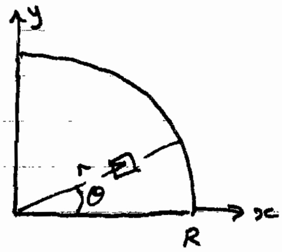
$$= \lambda^3 I_{xx} - (\lambda A)\lambda^2 \bar{y}^2$$

$$= \lambda^3 (I_{xx} - A\bar{y}^2) = \lambda^3 I_{xx}$$

$I_{y'y'}$, $I_{x'y'}$ similarly: $\longrightarrow$ $I_{y'y'} = \lambda I_{yy}$

$I_{x'y'} = \lambda^2 I_{xy}$.

Q1 (b). i)



$$A = \frac{1}{4}\pi R^2$$

Centroids:

$$\bar{x} = \frac{\int x\,dA}{\int dA} = \left(\frac{4}{\pi R^2}\right)\int_0^R \int_0^{\pi/2} r\cos\theta \,.\, r\,dr\,d\theta$$

$$= \left[\frac{4}{\pi R^2}\right]\int_0^R r^2 dr \int_0^{\pi/2} \cos\theta\, d\theta$$

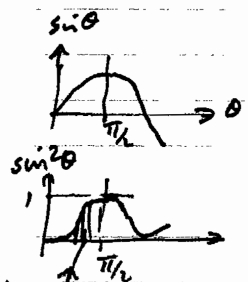
$$= \frac{4}{\pi R^2}\,\frac{R^3}{3}\left[\underbrace{+\sin \pi/2}_{=1} - \sin 0\right]$$

$$= \frac{4}{3\pi}\,.\,R = 0{\cdot}4244 R$$

$\bar{y} = \bar{x}$ by symmetry $= \frac{4}{3\pi}R$ also.

b) ii) $$I_{xx} = \int_A y^2\,dA = \int_0^{\pi/2}\int_0^R r^2 \sin^2\theta \; r\,dr\,d\theta$$

$$= \left[\int_0^{\pi/2} \sin^2\theta\, d\theta\right]\int_0^R r^3 dr = \frac{R^4}{4}\int_0^{\pi/2} \sin^2\theta\, d\theta$$



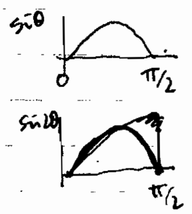
$$= \frac{\pi}{4}\,\frac{R^4}{4} = \frac{\pi R^4}{16}.$$

$\therefore = \frac{1}{2} \times \pi/2 =$ half area of $1 \times \pi/2$

$I_{yy} = I_{xx}$ similarly.

$$I_{xy} = \int xy\,dA = \int_0^{\pi/2}\int_0^R r\cos\theta \,.\, r\sin\theta \; r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \cos\theta\sin\theta\, d\theta \,.\, \int_0^R r^3 dr = \int_0^{\pi/2} \frac{(\sin 2\theta)}{2}\, d\theta \,.\, \frac{R^4}{4}$$



Q1 b) cont'd.

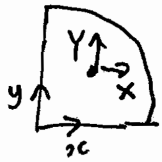
$$I_{xy} = \left[-\frac{\cos 2\theta}{4}\right]_0^{\pi/2} \frac{R^4}{4} = \frac{R^4}{16}\left[\underbrace{-\cos\pi + \cos 0}_{-(-1)+1=2}\right]$$

$$= \frac{R^4}{8}$$

Use matrix, or Mohr's circle. (Use former).

$$\underline{\underline{I}} = \begin{bmatrix} I_{xx} & I_{xy} \\ I_{xy} & I_{yy} \end{bmatrix} \quad \dots$$ No, not yet.

First, parallel axis to centroid.



$$I_{XX} = I_{xx} - A\bar{y}^2$$

$$= \frac{\pi R^4}{16} - \frac{\pi R^2}{4}\frac{16R^2}{9\pi^2} = R^4\left[\frac{\pi}{16} - \frac{4}{9\pi}\right] = 0.0549\,R^4$$

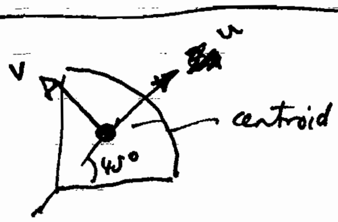
$I_{YY} = I_{XX}$ by symmetry

$$I_{XY} = I_{xy} - A\bar{x}\bar{y}$$

$$= \frac{R^4}{8} - \frac{\pi R^2}{4}\frac{16R^2}{9\pi^2} = R^4\left[\frac{1}{8} - \frac{4}{9\pi}\right] = -0.0165\,R^4$$

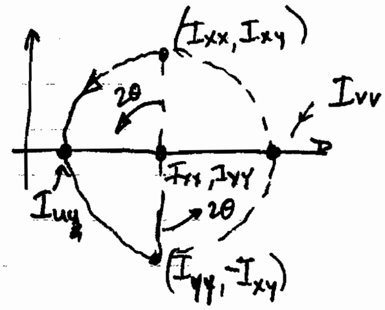
(= −ve).

Now: convert (i.e. rotate) to principal frame



Use Mohr's Circle:

NOT NOW NEEDED



$\therefore\ I_{uu} = I_{XX} - I_{XY}$

$I_{vv} = I_{YY} + I_{XY}$

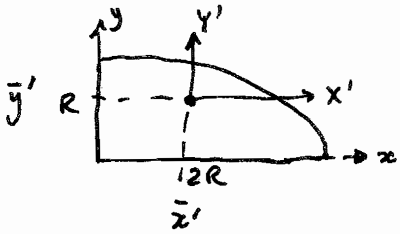
In general.

(Although noting in this case that I_{XY} is negative.)

$$\rightarrow\ I_{uu} = R^4\left[\pi/16 - \cancel{\frac{4}{9\pi}} - \frac{1}{8} + \cancel{\frac{4}{9\pi}}\right] = R^4\left[\pi/16 - 1/8\right]$$

$$I_{vv} = R^4\left[\pi/16 - \frac{4}{9\pi} + \frac{1}{8} - \frac{4}{9\pi}\right] = R^4\left[\pi/16 + \frac{1}{8} - \frac{8}{9\pi}\right]$$

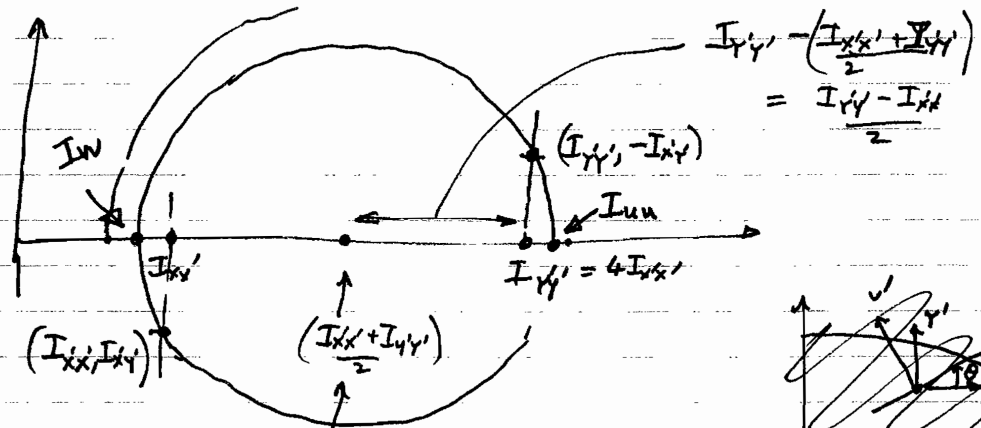
c)

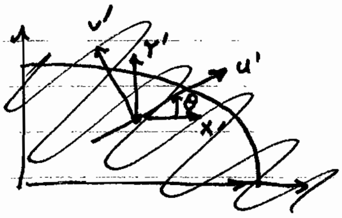


From part a) we have

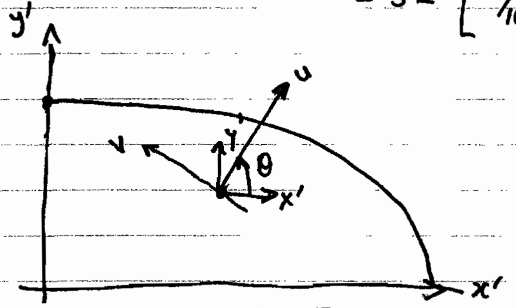
$$I_{x'x'} = 2\,I_{xx} = 2R^4\left[\pi/16 - 4/9\pi\right] = 0.11\,R^4$$
$$I_{y'y'} = 8\,I_{YY} = 8R^4\left[\pi/16 - 4/9\pi\right] = 0.44\,R^4$$
$$I_{x'y'} = 4\,I_{xy} = 4R^4\left[\tfrac{1}{8} - 4/9\pi\right] \quad (= -ve) \quad = -0.0659\,R^4$$

Use Mohr's Circle.





$$= 5R^4\left[\pi/16 - \frac{4}{9\pi}\right]$$



$$\text{Radius of Mohr's circle} = \sqrt{\left(I_{x'y'}\right)^2 + \left(\frac{I_{y'y'} - I_{x'x'}}{2}\right)^2}$$

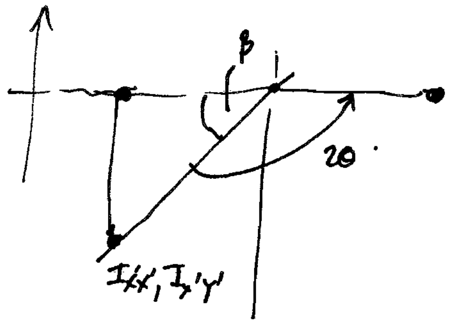
$$= R^4\left[16\left[\frac{1}{8} - \frac{4}{9\pi}\right]^2 + \underbrace{9}_{\left(\frac{8-2}{2}\right)^2}\left[\frac{\pi}{16} - \frac{4}{9\pi}\right]^2\right]^{1/2} =$$

$$\text{Centre of Mohr's circle} = \frac{I_{x'x'} + I_{y'y'}}{2} = 5R^4\left[\pi/16 - 4/9\pi\right]$$

$$\therefore \quad I_{uu} = R^4\left[5\left[\pi/16 - \frac{4}{9\pi}\right] + \sqrt{16\left(\frac{1}{8} - \frac{4}{9\pi}\right)^2 + 9\left(\frac{\pi}{16} - \frac{4}{9\pi}\right)^2}\right] = R^4\left[0.2744 + 0.1773\right] = R^4(0.4517)$$

$$I_{vv} = R^4\left[\quad '' \quad - \quad '' \quad\right] = R^4(0.2744 - 0.1773) = R^4(0.0971)$$

~~Q14~~ $\tan\theta = ?$



$$\tan\beta = \frac{-I_{x'y'}}{\text{Radius}} = \frac{0.0659\,R^4}{0.1773\,R^4}$$

$$\tan\beta = 0.3715$$

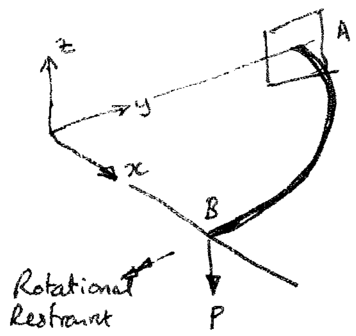
$$\therefore \quad \beta = 0.3557 \text{ radians.}$$

$$\therefore \; 2\theta = \pi - \beta = 2.7859$$

$$\therefore \; \theta = 1.3929 \text{ radians}$$

$$\therefore \quad \theta = \underline{\underline{79.8^\circ}}$$

3D4. Q2.



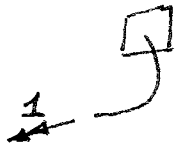
The system is 1 degree statically indeterminate.

Choose rotational restraint at B as the release.

∴ Consider System ⓪

statically determinate.

System ①



unit twisting moment at release, B.
Also statically determinate.

$$\text{Real system} = \text{System } ⓪ + \underset{\uparrow}{T_o} \text{ System } ①$$

The actual twisting moment at release B in the real system (in direction ↙)

Dot the real system with System ①. Virtual Work.

$$\sum W.\delta + \sum M.\theta = \int \frac{M.m}{EI}\, ds + \int \frac{T.t}{GJ}\, ds$$

Real system = Compatible = ⓪ + T_o ①

Equilibrium system = System ①.

The only external forces in System ① are/is the unit moment at B.

The conjugate rotation is the twist at B in the real system, but this is <u>zero</u>, since restrained.

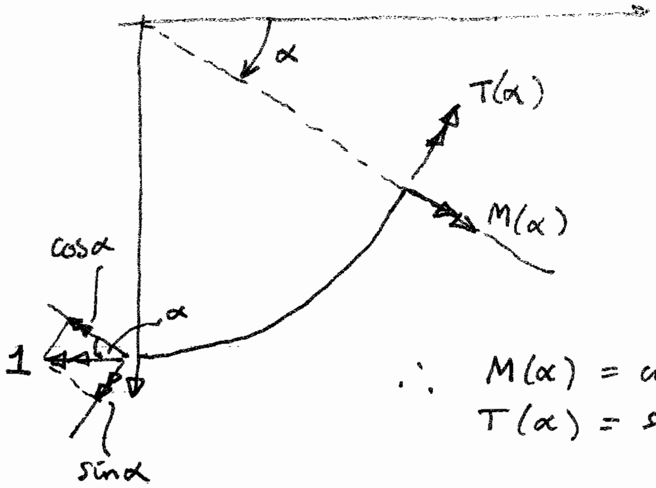
θ_{yy} at B in Real system.

$$\therefore \quad \text{V.W. L.H.S.} = 1.0 = 0$$

Torque at B in ①

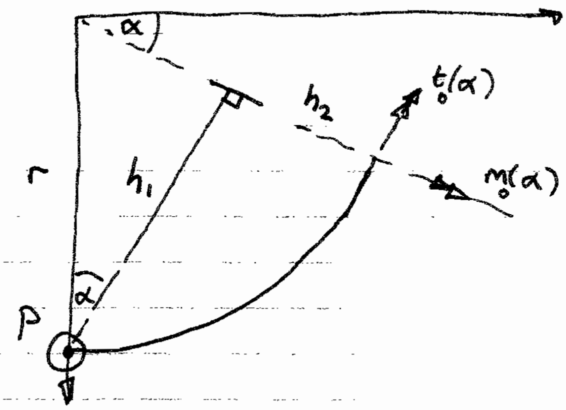
Q2 (cont'd)

Find the internal moments and torques $M(\alpha)$, $T(\alpha)$ in System ①.



$$\therefore \quad M(\alpha) = \cos\alpha$$
$$T(\alpha) = \sin\alpha$$

Find the internal moments and torques $m_0(\alpha)$, $t_0(\alpha)$ in System ⓪.



$$h_1 = r\cos\alpha$$
$$h_2 = r(1-\sin\alpha)$$

$$t_0(\alpha) = P \cdot h_2 = Pr(1-\sin\alpha)$$
$$m_0(\alpha) = -P \cdot h_1 = -Pr\cos\alpha$$

In real system, $$t(\alpha) = \underbrace{t_0(\alpha)}_{\text{system ⓪}} + T_0 \cdot \underbrace{T(\alpha)}_{\text{system ①}}$$
$$m(\alpha) = m_0(\alpha) + T_0 \cdot M(\alpha)$$

$$\therefore \quad t(\alpha) = Pr(1-\sin\alpha) + T_0 \cdot \sin\alpha$$
$$m(\alpha) = -Pr\cos\alpha + T_0 \cos\alpha$$

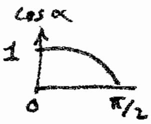
Q2 contd.

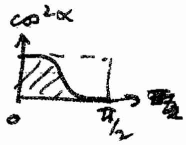
Virtual Work :–

$$0 = \int_0^{\pi/2} \frac{[\cos\alpha].\left[-P_r\cos\alpha + T_0\cos\alpha\right]}{EI}\, r\,d\alpha + \int_0^{\pi/2}\frac{[\sin\alpha]\left[P_r(1-\sin\alpha) + T_0\sin\alpha\right]}{GJ}\, r\,d\alpha$$

$$= \left(\frac{-P_r + T_0}{EI}\right) r\int_0^{\pi/2}\cos^2\alpha\,d\alpha + \frac{P_r^2}{GJ}\int_0^{\pi/2}\sin\alpha\,d\alpha + \left[\frac{-P_r+T_0}{GJ}\right] r\int_0^{\pi/2}\sin^2\alpha\,d\alpha$$

Now $\int_0^{\pi/2}\cos^2\alpha\,d\alpha$





$= \tfrac{1}{2}\cdot\tfrac{\pi}{2} = \pi/4$

$\int_0^{\pi/2}\sin^2\alpha\,d\alpha$ similarly $= \pi/4$

$\int_0^{\pi/2}\sin\alpha\,d\alpha = \left[-\cos\alpha\right]_0^{\pi/2} = [0]-[-1] = 1$

$$\therefore\ 0 = \left[\left[\frac{-P_r + T_0}{EI}\right]\frac{\pi}{4} + \frac{P_r}{GJ}.1 + \left[\frac{-P_r+T_0}{GJ}\right]\pi/4\right].r$$

$$\therefore\ P_r\left[+\frac{\pi}{4EI} - \frac{1}{GJ} + \frac{\pi}{4GJ}\right] = T_0\left[\frac{\pi}{4EI} + \frac{\pi}{4GJ}\right]$$

$$P_r\left[\frac{1}{EI} - \left(\frac{4}{\pi}-1\right)\frac{1}{GJ}\right] = T_0\left[\frac{1}{EI}+\frac{1}{GJ}\right]$$

$$\therefore\ P_r\left[GJ - \left(\tfrac{4}{\pi}-1\right)EI\right] = T_0\left[GJ + EI\right]$$

$$\therefore\ T_0 = P_r\,\frac{\left[GJ - \left(\tfrac{4}{\pi}-1\right)EI\right]}{GJ+EI}$$

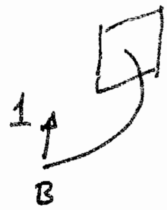
sign defined as +ve (via our definition of the release)

B

$$= P_r\left[1 - \frac{4}{\pi}\frac{1}{\left(1+\frac{GJ}{EI}\right)}\right]$$

3D4 Q2 b)

Define System (2) as the statically determinate system (released at B), and with a unit force in the vertical z direction at B.



= system (2).

Then dot system (2) with Real system = System (0) + T_a system (1).

LHS.

$$1 . z_B = \int M_2 . \frac{M}{EI} ds + \int T_2 . \frac{t}{GJ} ds.$$

Real = (0) + T_0 (1).

Equili = (2).

Determine $M_2(\alpha)$ and $T_2(\alpha)$ for system (2) by statics.

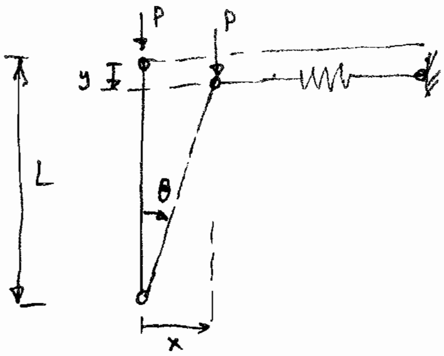
All other terms are known.

∴ Substitute, integrate RHS w.r.t α

$$\Rightarrow z_B = \int_0^{\pi/2} \frac{\sim}{EI} r . d\alpha + \int_0^{\pi/2} \frac{\dots}{GJ} r . d\alpha$$

3D4 2009. Q3.

a) i)



Total potential energy

$$\Pi = \underbrace{\tfrac{1}{2}kx^2}_{\text{strain energy}} - \underbrace{P.y}_{\text{W.D by ext. forces.}}$$

$$x = L\sin\theta$$
$$y = L(1-\cos\theta)$$

$$\Pi(\theta) = \tfrac{1}{2}kL^2\sin^2\theta - PL(1-\cos\theta)$$

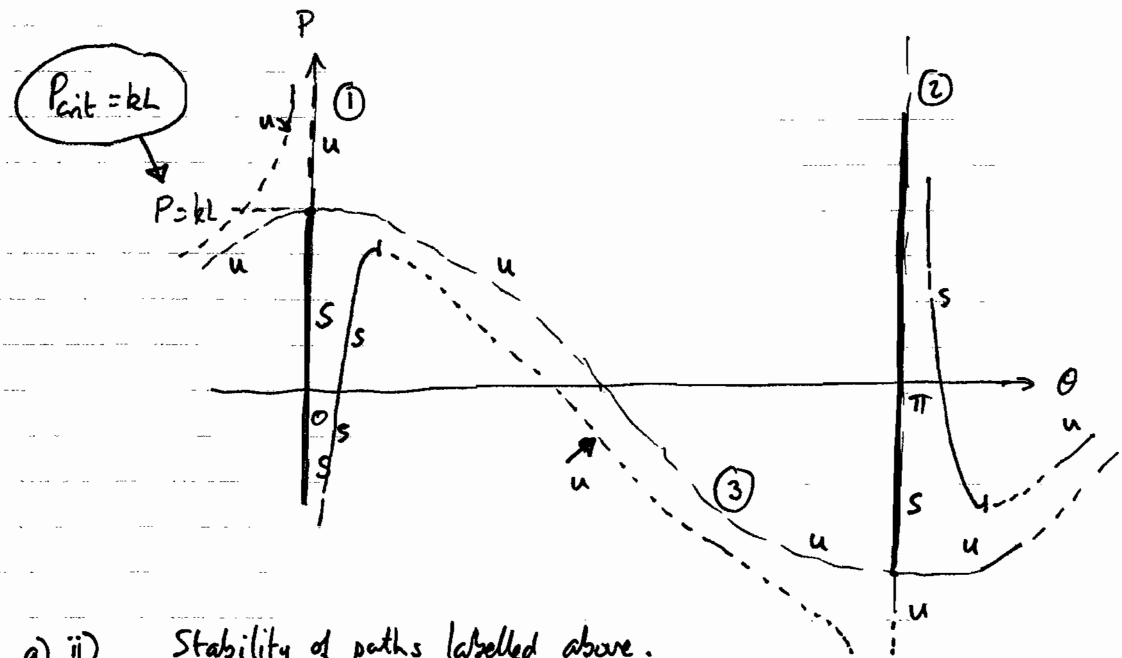
Equilib Solutions when $\frac{\partial \Pi}{\partial \theta} = 0$

$$0 = \cancel{\tfrac{1}{2}}kL^{\cancel{2}}\cancel{2}\sin\theta\cos\theta - P\cancel{L}\sin\theta$$

$$0 = \sin\theta\,(kL\cos\theta - P)$$

∴ Solutions $0 = \sin\theta \rightarrow \theta = 0$ or $\mp\pi$ (or $n\pi$) ①,②

also $kL\cos\theta - P = 0 \Rightarrow P = kL\cos\theta$. ③



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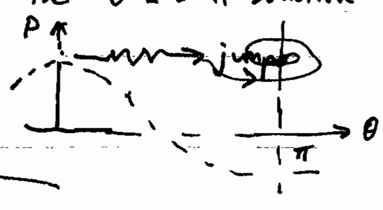
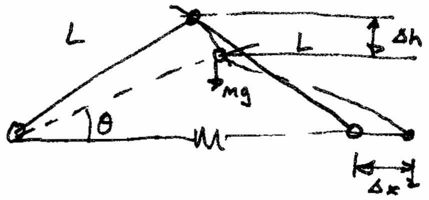
a) ii) Stability of paths labelled above.

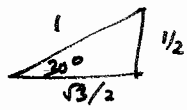
In perfect case, the rod remains perfectly vertical until $P = P_{crit}$ and then above this it jumps catastrophically to the $\theta = \pm\pi$ solution 4



a) iii) See above on diag. 4

Q3.b)





$\Pi(\theta)$ = Total P.E $= \frac{1}{2}k(\Delta x)^2 - m.g.\Delta h$

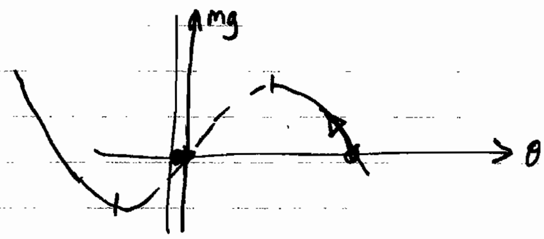
$$\Delta x = L_{new} - L_{orig}$$
$$= 2L\cos\theta - 2L\cos 30^\circ = 2L\cos\theta - \sqrt{3}L. = L(2\cos\theta - \sqrt{3})$$

$$\Delta h = L\sin 30^\circ - L\sin\theta = L/2 - L\sin\theta =$$

$$\therefore \quad \Pi(\theta) = \frac{1}{2}kL^2(2\cos\theta - \sqrt{3})^2 - mg\frac{L}{2}(1 - 2\sin\theta).$$

Equilib $\rightarrow$ $\quad \frac{\partial\Pi}{\partial\theta} = 0 = \cancel{\frac{1}{2}}kL^{\cancel{2}}.\cancel{2}.(-2\sin\theta)(2\cos\theta - \sqrt{3}) + mg\cancel{L}\cos\theta$

$$\therefore mg(\theta) = 2kL\frac{\sin\theta}{\cos\theta}(2\cos\theta - \sqrt{3})$$
$$= 4kL\sin\theta - 2\sqrt{3}kL\tan\theta$$



Instability when

$$\frac{\partial}{\partial\theta}(mg) = 0$$

$$\therefore 0 = +4kL\cos\theta - \frac{2\sqrt{3}kL}{\cos^2\theta}$$

$$\therefore \quad \cos^3\theta = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \quad (= \cos\theta_0) \text{ btw}$$

$$\therefore \cos\theta_{crit} = \sqrt[3]{\cos\theta_0} = \sqrt[3]{\frac{\sqrt{3}}{2}} = \frac{3^{1/6}}{2^{1/3}} = \sqrt[6]{\frac{3}{4}}$$

(Note: – independent of k!)

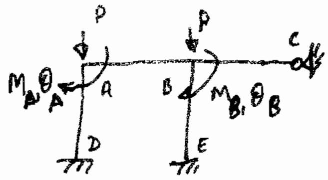
$\theta = 17.6^\circ$

[illegible]

3D4 2009.

Q4. a) Can get curvature without strain reversal if it bends during loading (etc). 4

b) i)



$$\begin{bmatrix} M_A \\ M_B \end{bmatrix} = k \begin{bmatrix} s+4 & 2 \\ 2 & s+7 \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix}$$

(being $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ from AB $\begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix}$ from AD, $\begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix}$ from BE

and $\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$ from BC).

Instability when $\begin{vmatrix} s+4 & 2 \\ 2 & s+7 \end{vmatrix} = 0 \Rightarrow s^2 + 4s + 7s + 28 - 4 = 0$

$$s^2 + 11s + 24 = 0$$

$$s = \frac{-11 \pm \sqrt{121 - 4(24)}}{2} = \frac{-11 \pm \sqrt{25}}{2}$$

$$= \frac{-16}{2} \text{ or } \frac{-6}{2}$$

$$= \underline{\underline{-8}} \text{ or } \underline{\underline{-3}}$$

↑ critical.

from Fig 7 $\rightarrow$ $\frac{P}{P_E} \approx \underline{\underline{2.7}}$. 6

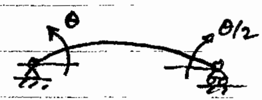
b) ii)

Put $s = -3$

$$(s+4)\theta_A + 2\theta_B = 0$$

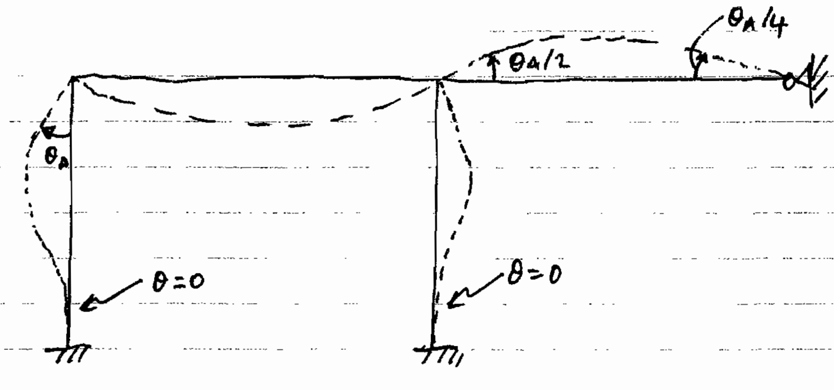
$$\Rightarrow \quad \theta_A + 2\theta_B = 0 \quad \Rightarrow \quad \underline{\theta_B = -\theta_A/2}$$

Also



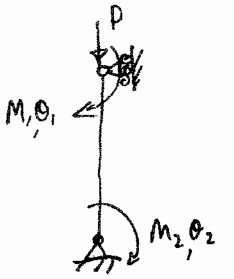
True even if not on roller.

$$\therefore \quad \theta_C = -\frac{\theta_B}{2} = \frac{\theta_A}{4}.$$



4

Q4.b)iii)



$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = k \begin{bmatrix} s & sc \\ sc & s \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

but base pinned $\Rightarrow$ $M_2 = 0$

$\therefore$ Last eqn $\Rightarrow$ $sc\,\theta_1 + s\,\theta_2 = 0 \rightarrow \theta_2 = -c\,\theta_1$

First eqn $\Rightarrow$
$$M_1 = k\,s\,\theta_1 + k\,sc\,\theta_2$$
$$= ks\,[\theta_1 + c(-c\theta_1)]$$
$$M_1 = ks(1-c^2)\,\theta_1$$

$\therefore$ Matrix becomes
$$\begin{bmatrix} s(1-c^2)+4 & 2 \\ 2 & s(1-c^2)+7 \end{bmatrix}$$

Det $= 0$ $\Rightarrow$ $s(1-c^2) = -8$ or -3 (again)

From Fig 7. $\Rightarrow$ $P/P_E \approx \underline{\underline{1{\cdot}4}}$ for $s(1-c^2) = -3$.

Put $s(1-c^2) = -3 \rightarrow \theta_A + 2\theta_B = 0$ again

$\therefore [\theta_A : \theta_B : \theta_C] = [1 : -1/2 : 1/4]$ again.

From "$\theta_2 = -c\theta_1$" for columns $\theta_D = -c\,\theta_A$

and for $P/P_E \approx 1{\cdot}4 \rightarrow c = 1{\cdot}66$
$$\theta_D = -1{\cdot}66\,\theta_A$$
and $\theta_E = -1{\cdot}66\,\theta_B = 0{\cdot}83\,\theta_A$

$\therefore [\theta_A : \theta_B : \theta_C : \theta_D : \theta_E] = [1 : -1/2 : \frac{1}{4} : -1{\cdot}66 : 0{\cdot}83]$