Dr F Cirak

# 3D7: Finite Element Method
## — Crib for 2009 —

1. (a)

i.

$$N_1 = \frac{x^2}{2} - \frac{x}{2} \qquad \frac{dN_1}{dx} = x - \frac{1}{2}$$
$$N_2 = \frac{x^2}{2} + \frac{x}{2} \qquad \frac{dN_2}{dx} = x + \frac{1}{2}$$
$$N_3 = -x^2 + 1 \qquad \frac{dN_3}{dx} = -2x$$

$$\boldsymbol{B} = \begin{bmatrix} x - \frac{1}{2} & x + \frac{1}{2} & -2x \end{bmatrix}$$

ii.

$$\boldsymbol{k}_e = \int_{-1}^{+1} EA\boldsymbol{B}^T\boldsymbol{B}\,dx$$

The integrand is a fourth order polynomial; three integration points required for exact integration (see data sheet).

$$\frac{1}{E_0A_0}k_{1j} = \sum_{G=1}^{3}(1+x^2)(x-\frac{1}{2})\frac{dN_j}{dx}w_G \quad \text{(summation over three integration points)}$$

$$\begin{aligned}\frac{1}{E_0A_0}k_{11} &= (1+\frac{3}{5})(-\sqrt{\frac{3}{5}}-\frac{1}{2})(-\sqrt{\frac{3}{5}}-\frac{1}{2})\frac{5}{9} \\ &+ (1+0)(-\frac{1}{2})(-\frac{1}{2})\frac{8}{9} \\ &+ (1+\frac{3}{5})(\sqrt{\frac{3}{5}}-\frac{1}{2})(\sqrt{\frac{3}{5}}-\frac{1}{2})\frac{5}{9} \\ &= 1.733\end{aligned}$$

$$\begin{aligned}\frac{1}{E_0A_0}k_{12} &= (1+\frac{3}{5})(-\sqrt{\frac{3}{5}}-\frac{1}{2})(-\sqrt{\frac{3}{5}}+\frac{1}{2})\frac{5}{9} \\ &+ (1+0)(-\frac{1}{2})(\frac{1}{2})\frac{8}{9} \\ &+ (1+\frac{3}{5})(\sqrt{\frac{3}{5}}-\frac{1}{2})(\sqrt{\frac{3}{5}}+\frac{1}{2})\frac{5}{9} \\ &= 4\cdot 10^{-1}\end{aligned}$$

$$
\begin{aligned}
\frac{1}{E_0 A_0} k_{13} &= (1+\frac{3}{5})(-\sqrt{\frac{3}{5}}-\frac{1}{2})(2\sqrt{\frac{3}{5}})\frac{5}{9} \\
&+0 \\
&+(1+\frac{3}{5})(\sqrt{\frac{3}{5}}-\frac{1}{2})(-2\sqrt{\frac{3}{5}})\frac{5}{9} \\
&= -2.133
\end{aligned}
$$

(b) i.

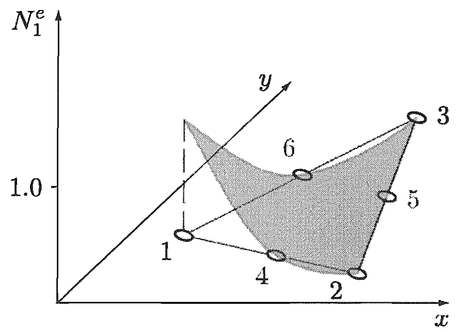
$n$ quadratic elements have $n$ mid-nodes and $n+1$ end nodes. This means that there are $2n+1$ total nodes. The stiffness matrix dimension is approximately $2n \times 2n$, or for 10000 elements there are $4 \times 10^8$ entries. Hence, the required memory is $4 \times 10^8 \times 8\,\text{bytes} = 3.2 \times 10^9$ bytes, which is approximately $2.98\,GB$.

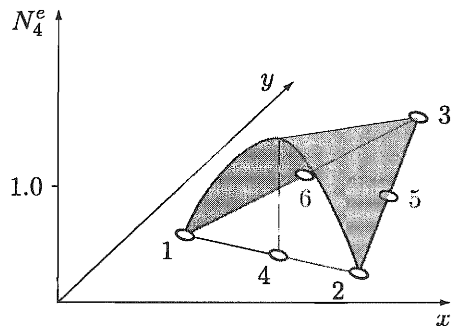
ii. Sparse storage:
- mid nodes have 2 neighbors $\rightarrow$ $(2+1)n$ entries
- end nodes have 4 neighbors $\rightarrow$ $(4+1)n$ entries

This means that there are $8n$ non-zero entries. Hence, the required memory is $8 \times 10^4 \times 8\,\text{bytes} = 6.4 \times 10^5$ bytes, which is approximately $625\,kB$

2. (a)

i.




From data sheet:

$$
\begin{aligned}
N_1(\xi,\,\eta) &= 2(1-\xi-\eta)^2 - (1-\xi-\eta) \\
N_4(\xi,\,\eta) &= 4\xi(1-\xi-\eta)
\end{aligned}
$$

Scaling of the shape functions to the $xy-$ coordinate system:

$$
\xi = \frac{x}{L} \qquad \eta = \frac{y}{L}
$$

$$
\begin{aligned}
N_1(x,\,y) &= 2(1-\frac{x}{L}-\frac{y}{L})^2 - (1-\frac{x}{L}-\frac{y}{L}) \\
N_2(x,\,y) &= 4\frac{x}{L}(1-\frac{x}{L}-\frac{y}{L})
\end{aligned}
$$

ii.

$$
\boldsymbol{f}^e = -\int_{\Gamma_q^e} \boldsymbol{N}^{e^T} \overline{q}\, d\Gamma
$$

At node 1: element source vector component is zero because of the opposite signs of the fluxes along $1-2$ and $1-3$.

At node 4:

$$-\bar{q}\int_0^L 4\frac{x}{L}(1-\frac{x}{L})\,dx = -\frac{4\bar{q}}{L}\int_0^L x - \frac{x^2}{L}\,dx$$
$$= -\frac{2}{3}\bar{q}L$$

At node 2:

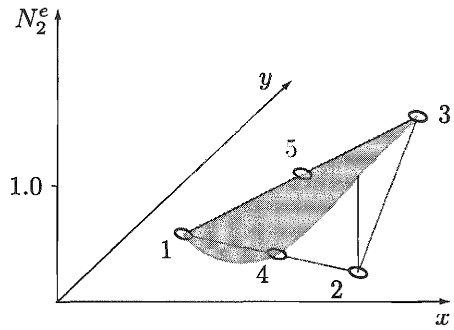
$$-\bar{q}\int_0^L \left(2(\frac{x}{L})^2 - \frac{x}{L}\right)\,dx = -\frac{\bar{q}}{6}$$

Using these results and considering the similarity of the line integrals along $1-2$ and $1-3$ the element source vector is:

$$f^e = \begin{bmatrix}0 & -\frac{1}{6}\bar{q}L & \frac{1}{6}\bar{q}L & -\frac{2}{3}\bar{q}L & 0 & \frac{2}{3}\bar{q}L\end{bmatrix}^T$$

(b)

i.




ii. The shape function $N_1$ for the regular quadratic triangular element is zero along the edge 2-3 and zero at the nodes 4 and 5. Hence, the shape function $N_1$ of the usual quadratic triangular element is also the shape function for node 1 of the transition element.

3. (a) Shape function values at the quadrature point $(1/\sqrt{3},\, 1/\sqrt{3})$

$$N_1(1/\sqrt{3},\, 1/\sqrt{3}) = 0.0893 \qquad N_2(1/\sqrt{3},\, 1/\sqrt{3}) = 0.1667$$
$$N_3(1/\sqrt{3},\, 1/\sqrt{3}) = 0.6220 \qquad N_4(1/\sqrt{3},\, 1/\sqrt{3}) = 0.1667$$

Coordinates of the point $P$ in the isoparametric element

$$x = 0 + 0.1667 \times 2.0 + 0.6220 \times 1.0 + 0 = 0.9554$$
$$y = 0 + 0 + 0.6220 \times 3.0 + 0.1667 \times 2.0 = 2.1994$$

(b)

Shape function derivatives at the quadrature point $(1/\sqrt{3},\, 1/\sqrt{3})$

$$\frac{\partial N_1}{\partial \xi} = -0.1057 \qquad \frac{\partial N_1}{\partial \eta} = -0.1057$$
$$\frac{\partial N_2}{\partial \xi} = 0.1057 \qquad \frac{\partial N_2}{\partial \eta} = -0.3943$$
$$\frac{\partial N_3}{\partial \xi} = 0.3943 \qquad \frac{\partial N_3}{\partial \eta} = 0.3943$$
$$\frac{\partial N_4}{\partial \xi} = -0.3943 \qquad \frac{\partial N_4}{\partial \eta} = 0.1057$$

$$\frac{\partial x}{\partial \xi} = 0 + 0.1057 \times 2 + 0.3943 \times 1.0 + 0 = 0.6057$$
$$\frac{\partial x}{\partial \eta} = 0 - 0.3943 \times 2 + 0.3943 \times 1.0 + 0 = -0.3943$$
$$\frac{\partial y}{\partial \xi} = 0 + 0 + 0.3943 \times 3 - 0.3943 \times 2 = 0.3943$$
$$\frac{\partial y}{\partial \eta} = 0 + 0 + 0.3943 \times 3 + 0.1057 \times 2 = 1.3943$$

$$\Rightarrow \quad \boldsymbol{J} = \begin{bmatrix} 0.6057 & 0.3943 \\ -0.3943 & 1.3943 \end{bmatrix}$$

(c)

$$\begin{aligned} \epsilon_{xx} &= \frac{\partial u_x}{\partial x} \\ &= \frac{\partial u_x}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial x} \end{aligned}$$

To compute $\frac{\partial \xi}{\partial x}$ and $\frac{\partial \eta}{\partial x}$, the inverse of the Jacobian is needed

$$\boldsymbol{J}^{-1} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} = \begin{bmatrix} 1.3943 & -0.3943 \\ 0.3943 & 0.6057 \end{bmatrix}$$

Furthermore, only the shape function of node 3 is relevant

$$\epsilon_{xx} = 0.3943 \times 0.01 \times 1.3943 + 0.3943 \times 0.01 \times (-0.3943) = 3.943 \cdot 10^{-3}$$

$$\begin{aligned} \epsilon_{yy} &= \frac{\partial u_y}{\partial y} \\ &= \frac{\partial u_y}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u_y}{\partial \eta}\frac{\partial \eta}{\partial y} \\ &= 0.3943 \times 0.02 \times 0.3943 + 0.3943 \times 0.02 \times 0.6057 = 7.886 \cdot 10^{-3} \end{aligned}$$

$$\epsilon_{xy} = \frac{1}{2}(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x})$$

with

$$\frac{\partial u_x}{\partial y} = \frac{\partial u_x}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial y} = 3.943 \cdot 10^{-3}$$
$$\frac{\partial u_y}{\partial x} = \frac{\partial u_y}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u_y}{\partial \eta}\frac{\partial \eta}{\partial x} = 7.886 \cdot 10^{-3}$$

$$\Rightarrow \epsilon_{xy} = 5.915 \cdot 10^{-3}$$

4. (a)

i. Two degrees of freedom per node for Hermitian shape functions. This means that the dimension of the global mass matrix is $6 \times 6$.

ii.

$$\begin{bmatrix} \rho l_1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho(l_1/2 + l_2/2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho l_2/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The problem is that with the simple mass lumping, we have no mass associated with the rotation degrees of freedom. This may lead to an instability. Row sum approach is more appropriate.

iii. Lumped mass matrices are useful in combination with explicit time integration schemes. Solving a system of equations can be avoided.

(b)

i.

$$\boldsymbol{a}_{n+1} - \boldsymbol{a}_{n-1} = 2\Delta t \dot{\boldsymbol{a}}_n$$
$$\Rightarrow \dot{\boldsymbol{a}}_n = \frac{1}{2\Delta t}(\boldsymbol{a}_{n+1} - \boldsymbol{a}_{n-1})$$

Pose the matrix problem at $t_n$

$$\boldsymbol{M}\dot{\boldsymbol{a}}_n + \boldsymbol{K}\boldsymbol{a}_n = \boldsymbol{f}_n$$
$$\frac{1}{2\Delta t}\boldsymbol{M}(\boldsymbol{a}_{n+1} - \boldsymbol{a}_{n-1}) + \boldsymbol{K}\boldsymbol{a}_n = \boldsymbol{f}_n$$
$$\boldsymbol{M}\boldsymbol{a}_{n+1} = 2\Delta t \boldsymbol{f}_n - 2\Delta t \boldsymbol{K}\boldsymbol{a}_n + \boldsymbol{M}\boldsymbol{a}_{n-1}$$

ii. The scheme is explicit as it involves $\boldsymbol{a}_{n+1}$ but only $\dot{\boldsymbol{a}}_n$. It is not self starting as it requires information at $\boldsymbol{a}_{n-1}$.

# The list of numerical answers for 3D7

1. (a)

$$k_{11} = 1.733 E_0 A_0$$
$$k_{12} = 0.4 E_0 A_0$$
$$k_{13} = -2.133 E_0 A_0$$

   (b) (i) $3.2 \cdot 10^9$ bytes; (ii) $6.4 \cdot 10^5$ bytes

2. (a) (ii)

$$\boldsymbol{f}^e = \begin{bmatrix} 0 & -\frac{1}{6}\overline{q}L & \frac{1}{6}\overline{q}L & -\frac{2}{3}\overline{q}L & 0 & \frac{2}{3}\overline{q}L \end{bmatrix}^T$$

3. (a) $x = 0.9554$ $y = 2.1994$

   (b)

$$\boldsymbol{J} = \begin{bmatrix} 0.6057 & 0.3943 \\ -0.3943 & 1.3943 \end{bmatrix}$$

   (c) $\epsilon_{xx} = 3.943 \cdot 10^{-3}$, $\epsilon_{yy} = 7.886 \cdot 10^{-3}$, $\epsilon_{xy} = 5.915 \cdot 10^{-3}$