2009 PART IIA 3F1 – SIGNALS AND SYSTEMS Dr J. Gonçalves

1 (a) The z-transform of $\{v_k\}$ is given by

$$\begin{aligned} V(z) &= \sum_{k=0}^{\infty} v_k z^{-k} \\ &= \left(f_0 + f_1 z^{-1} + \cdots + f_{n-1} z^{-(n-1)}\right)\left(1 + z^{-n} + z^{-2n} + \cdots\right) \\ &= \frac{F(z)}{1 - z^{-n}} \end{aligned}$$

[20%]

(b) (i) Taking z-transforms and assuming zero initial conditions gives $Y(z) = \beta z^{-n} Y(z) + U(z) - z^{-n} U(z)$, or

$$\frac{Y(z)}{U(z)} = \frac{1 - z^{-n}}{1 - \beta z^{-n}}$$

[20%]

(ii) zeros: n^{th} roots of 1; poles: n^{th} roots of β. For $n = 1$ zero at 1 and pole at β. For $n = 2$ zeros at ± 1 and poles at $\pm\sqrt{\beta}$. Pole-zero diagrams can be found in Fig. 1. [20%]

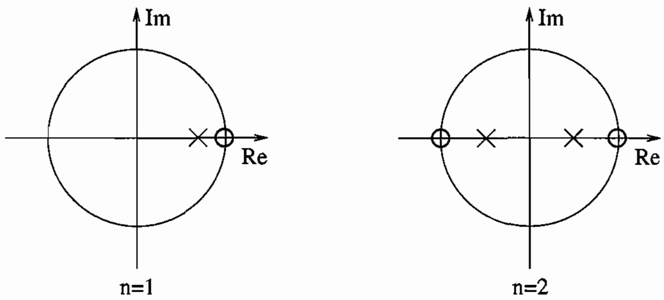


Fig. 1

(iii) We have that

$$\begin{aligned} Y(z) &= \frac{F(z)}{1 - z^{-n}} \cdot \frac{1 - z^{-n}}{1 - \beta z^{-n}} \\ &= \frac{F(z)}{1 - \beta z^{-n}} \\ &= \left(f_0 + f_1 z^{-1} + \cdots + f_{n-1} z^{-(n-1)}\right)\left(1 + \beta z^{-n} + \beta^2 z^{-2n} + \cdots\right) \end{aligned}$$

(TURN OVER for continuation of Question 1

A sketch of the response of the filter can be found in Fig. 2. [20%]

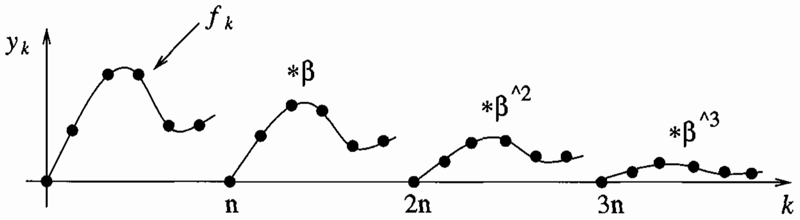


Fig. 2

(iv) A step can be written as $\frac{F(z)}{1-z^{-n}}$ where $F(z) = 1+z^{-1}+\cdots+z^{-n+1}$, i.e. pick $f_k = 1$, $k = 0,...,n-1$. Hence, from the previous part the step response if given by

$$\frac{F(z)}{1-\beta z^{-n}} = \left(1+z^{-1}+\cdots+z^{-n+1)}\right)\left(1+\beta z^{-n}+\beta^2 z^{-2n}+\cdots\right)$$

A sketch of the step response of the filter can be found in Fig. 3. [20%]

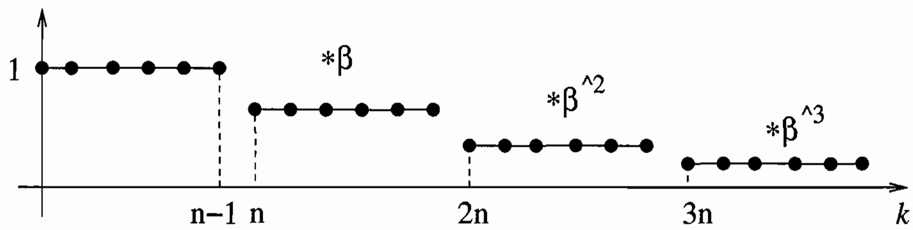


Fig. 3

2 (a) (i) The closed-loop poles are the roots of

$$1+k\frac{\alpha}{\beta z-1}=0$$

or the roots of $\beta z - 1 + k\alpha = 0$. For stability, we require the closed-pole $z = \frac{1-k\alpha}{\beta}$ to lie inside the unit disc, or

$$-1 < \frac{k\alpha - 1}{\beta} < 1$$

Since $\alpha > 0$ and $\beta > 0$ this is equivalent to

$$\frac{1-\beta}{\alpha} < k < \frac{1+\beta}{\alpha}$$

[15%]

(ii) $E(z)$ is given by

$$E(z) = \frac{1}{1+k\frac{\alpha}{\beta z-1}} \cdot U(z) = \frac{\beta z-1}{\beta z-1+k\alpha} \cdot \frac{z}{z-1}$$

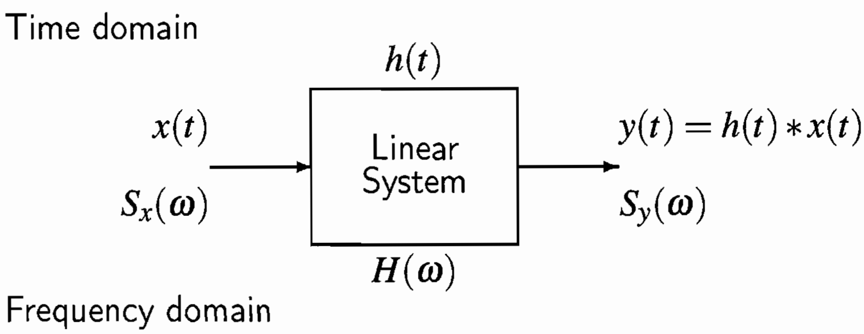
Thus, the Final Value Theorem (FVT) applies and

$$\lim_{k\to\infty} e_k = \lim_{z\to 1}(z-1)E(z) = \lim_{z\to 1}\frac{(\beta z-1)z}{\beta z-1+k\alpha} = \frac{\beta-1}{\beta-1+k\alpha}$$

[25%]

(iii) From part (a)(ii) we need $\beta = 1$ and the system to be stable. Thus, from part (a)(i), we require $0 < k < 2$. [10%]

(b) (i) Consider the linear system:



Block diagram of a linear system with a random input signal, $x(t)$.

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Let the linear system with input $x(t)$ and output $y(t)$ have an impulse response $h(t)$, so

$$y(t) = h(t) * x(t) = \int h(\beta)\, x(t-\beta)\, d\beta$$

Then the ACF of $y(t)$ is

$$\begin{aligned}
r_{yy}(t_1,t_2) &= E\,[y(t_1)\; y(t_2)] \\
&= E\left[\left(\int h(\beta_1)x(t_1-\beta_1)\, d\beta_1\right)\left(\int h(\beta_2)x(t_2-\beta_2)\, d\beta_2\right)\right] \\
&= E\left[\int\!\!\int h(\beta_1)\, h(\beta_2)\, x(t_1-\beta_1)\, x(t_2-\beta_2)\, d\beta_1\, d\beta_2\right] \\
&= \int\!\!\int h(\beta_1)\, h(\beta_2)\, E\,[x(t_1-\beta_1)\, x(t_2-\beta_2)]\; d\beta_1\, d\beta_2 \\
&= \int\!\!\int h(\beta_1)\, h(\beta_2)\, r_{xx}(t_1-\beta_1, t_2-\beta_2)\, d\beta_1\, d\beta_2
\end{aligned}$$

Since x is wide-sense stationary, then we substitute $\tau = t_2 - t_1$ and $t = t_1$ to get

$$\begin{aligned}
r_{yy}(\tau) &= E\,[y(t)\; y(t+\tau)] \\
&= \int\!\!\int h(\beta_1)\, h(\beta_2)\, r_{xx}(\tau+\beta_1-\beta_2)\, d\beta_1\, d\beta_2
\end{aligned}$$

If x is white, then $r_{xx}(\tau) = \delta(\tau)$. Hence

$$\begin{aligned}
r_{yy}(\tau) &= \int\!\!\int h(\beta_1)\, h(\beta_2)\, \delta(\tau+\beta_1-\beta_2)\, d\beta_2\, d\beta_1 \\
&= \int h(\beta_1)\, h(\tau+\beta_1)\, d\beta_1 \;=\; h(\tau) * h(-\tau)
\end{aligned}$$

Taking Fourier transforms:

$$\begin{aligned}
S_y(\omega) &= \mathrm{FT}\{r_{yy}(\tau)\} \\
&= \int\left(\int h(\beta_1)\, h(\tau+\beta_1)\, d\beta_1\right) e^{-j\omega\tau}\, d\tau \\
&= \int h(\beta_1)\left(\int h(\tau+\beta_1)\, e^{-j\omega\tau}\, d\tau\right) d\beta_1 \\
&= \int h(\beta_1)\left(\int h(\lambda)\, e^{-j\omega(\lambda-\beta_1)}\, d\lambda\right) d\beta_1 \\
&= \left(\int h(\beta_1)\, e^{j\omega\beta_1}\, d\beta_1\right)\left(\int h(\lambda)\, e^{-j\omega\lambda}\, d\lambda\right) \\
&= H^*(\omega)\, H(\omega) \;=\; |H(\omega)|^2 \qquad \text{where } H(\omega) \;=\; \mathrm{FT}\{h(t)\}
\end{aligned}$$

(cont.

Alternatively, we could have used properties of the Fourier transforms: convolution in time is multiplication in frequency and the Fourier transform of $h(-t)$ is $H^*(j\omega)$. Hence the PSD of y =the power gain $|H|^2$ of the system at frequency ω. [35%]

(ii) If a large and important system is subject to random nearly-white perturbations (e.g. a power plant subject to random load fluctuations), we may measure $r_{yy}(\tau)$, transform it to $S_y(\omega)$, and hence obtain

$$|H(\omega)| \approx \sqrt{S_y(\omega)}$$

Hence we may measure the system frequency response without taking the plant off line. The approximation comes from the fact that white noise cannot be realised in practice and can be approximated by nearly-white noise. [15%]

3 (a) Random variable Z is the sum of two independent random variables X and Y:

$$Z = X + Y$$

where X has pdf $f_X(x)$ and Y has pdf $f_Y(y)$.

We can write the joint pdf for Z and X as:

$$f_{Z,X}(z,x) = f_{Z|X}(z|x)\, f_X(x)$$

For X and Y independent, the event 'Z takes the value z conditional upon $X = x$' is equivalent to Y taking a value $z - x$ (since $Y = Z - X$). Hence

$$f_{Z|X}(z|x) = f_Y(z-x)$$

Now $f_Z(z)$ may be obtained using the *Marginal Probability* formula. Hence

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_{Z|X}(z|x)\, f_X(x)\, dx \\ &= \int_{-\infty}^{\infty} f_Y(z-x)\, f_X(x)\, dx \\ &= f_Y * f_X \end{aligned}$$

which is the convolution of f_X with f_Y. [25%]

(b) The characteristic function of a pdf is defined as:

$$\Phi_X(u) = E[e^{juX}] = \int_{-\infty}^{\infty} e^{jux} f_X(x)\, dx = F(-u)$$

where $F(\omega) = \int_{-\infty}^{\infty} e^{-j\omega x} f_X(x)\, dx$ is the Fourier Transform of the pdf.

Using the previous result for $f_Z(z)$:

$$\begin{aligned} \Phi_Z(u) &= \int_{-\infty}^{\infty} e^{juz} f_Z(z)\, dz \\ &= \int_{-\infty}^{\infty} e^{juz} \int_{-\infty}^{\infty} f_Y(z-x)\, f_X(x)\, dx\, dz \\ &= \int_{-\infty}^{\infty} f_X(x) \int_{-\infty}^{\infty} f_Y(z-x)\, e^{juz}\, dz\, dx \\ &= \int_{-\infty}^{\infty} f_X(x) \int_{-\infty}^{\infty} f_Y(y)\, e^{juy}\, dy \;\; e^{jux}\, dx \\ &= \int_{-\infty}^{\infty} f_X(x)\, \Phi_Y(u)\, e^{jux}\, dx \\ &= \Phi_X(u)\, .\, \Phi_Y(u) \end{aligned}$$

[25%]

(cont.

(c) The characteristic function of X is given by

$$\begin{aligned}
\Phi_X(u) &= E[e^{jux}] \\
&= \int_{-\infty}^{\infty} f(x) e^{jux} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{2x_0} e^{-|x|/x_0} e^{jux} dx \\
&= \frac{1}{2x_0} \int_{-\infty}^{0} e^{x/x_0} e^{jux} dx + \frac{1}{2x_0} \int_{0}^{\infty} e^{-x/x_0} e^{jux} dx \\
&= \frac{1}{2x_0\left(\frac{1}{x_0} + ju\right)} \left[e^{x\left(\frac{1}{x_0}+ju\right)} \right]_{-\infty}^{0} + \frac{1}{2x_0\left(\frac{-1}{x_0} + ju\right)} \left[e^{x\left(\frac{-1}{x_0}+ju\right)} \right]_{0}^{\infty} \\
&= \frac{1}{2(1+jux_0)} - \frac{1}{2(-1+jux_0)} \\
&= \frac{1}{1+u^2x_0^2}
\end{aligned}$$

(Note: $\Phi_X(0) = 1$, so pdf is valid.)

Hence,

$$\begin{aligned}
\Phi_Z(u) &= \Phi_X(u) \cdot \Phi_Y(u) \\
&= \frac{1}{\left(1+u^2x_0^2\right)\left(1+u^2y_0^2\right)}
\end{aligned}$$

[25%]

(d) To prove the central limit theorem, we need first to show that for a Gaussian pdf of zero mean $\Phi_X(u) = \exp^{-\sigma^2 u^2/2}$. Then, when N independent (not necessarily Gaussian) identical pdfs are scaled by $1/\sqrt{N}$ and added together, the characteristic function of the result is

$$\Phi_X(u) = \prod_{i=1}^{N} \Phi\left(\frac{u}{\sqrt{N}}\right) = \Phi^N\left(\frac{u}{\sqrt{N}}\right)$$

where $\Phi\left(\frac{u}{\sqrt{N}}\right)$ is the characteristic function of each of the separate input pdfs, after scaling by $1/\sqrt{N}$. Assuming zero mean implies that $\Phi'(0) = 0$. Hence, taking logs:

$$\begin{aligned}
\log \Phi_X(u) &= N \log \Phi\left(\frac{u}{\sqrt{N}}\right) \\
&= N \log\left(1 + \frac{u^2}{2N}\Phi''(0) + \text{terms of order } N^{-3/2} \text{ or smaller}\right) \\
&= \frac{u^2}{2}\Phi''(0) + \text{terms of order } N^{-1/2} \text{ or smaller}
\end{aligned}$$

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As $N \to \infty$

$$\log \Phi_X(u) \to -\frac{u^2\sigma^2}{2}$$

since $\Phi''(0) = -\sigma^2$. Thus,

$$\Phi_X(u) \to \exp^{-u^2\sigma^2/2}$$

which is the characteristic function of a Gaussian of variance σ^2. [25%]

4 (a) Mutual information is defined as:

$$I(X;Y) = H(X) - H(X|Y)$$

Based on this, we may obtain the following general expression for conditional entropy and hence mutual information:

$$\begin{aligned} H(X|Y) &= -\sum_{j=1}^{N_Y} p(y_j) \sum_{i=1}^{N_X} p(x_i|y_j) \log_2 p(x_i|y_j) \\ &= -\sum_j \sum_i p(x_i,y_j) \log_2 p(x_i|y_j) \quad \text{since } p(x,y) = p(x|y)\ p(y) \\ I(X;Y) &= -\sum_i p(x_i) \log_2 p(x_i) + \sum_j \sum_i p(x_i,y_j) \log_2 p(x_i|y_j) \\ &= \sum_j \sum_i p(x_i,y_j)[\log_2 p(x_i|y_j) - \log_2 p(x_i)] \quad \text{since } \sum_j p(x,y_j) = p(x) \\ &= \sum_j \sum_i p(x_i,y_j) \log_2 \left(\frac{p(x_i|y_j)}{p(x_i)} \right) \end{aligned}$$

$I(Y;X)$ can also be obtained in the same way to give:

$$I(Y;X) = \sum_i \sum_j p(x_i,y_j) \log_2 \left(\frac{p(y_j|x_i)}{p(y_j)} \right)$$

Using Bayes rule,

$$p(x_i|y_j)\ p(y_j) = p(x_i,y_j) = p(y_j|x_i)\ p(x_i)$$

and so

$$\frac{p(x_i|y_j)}{p(x_i)} = \frac{p(y_j|x_i)}{p(y_j)}$$

Hence

$$I(Y;X) = I(X;Y)$$

i.e. X provides the same amount of information about Y as Y does about X. [35%]

(b) From above

$$\begin{aligned} I(Y;X) &= H(Y) - H(Y|X) \\ &= H(Y) - \beta H([0.15 \quad 0.85]) - (1-\beta) H([0.85 \quad 0.15]) \\ &= H(Y) - H([0.15 \quad 0.85]) \end{aligned}$$

Now, $P(x=1) = \beta$ and $P(x=-1) = 1-\beta$. Thus, $P(y=1) = P(y=1|x=1) \cdot P(x=1) + P(y=1|x=-1) \cdot P(x=-1) = 0.85\beta + 0.15(1-\beta) = 0.15 + 0.7\beta$, and $P(y=-1) =$

(TURN OVER for continuation of Question 4

$1-P(y=1)=0.85-0.7\beta$. Therefore, $H(y)=-(0.15+0.7\beta)\log_2(0.15+0.7\beta)-(0.85-0.7\beta)\log_2(0.85-0.7\beta)$. Finally,

$$
\begin{aligned}
I(Y;X) &= -(0.15+0.7\beta)\log_2(0.15+0.7\beta)-(0.85-0.7\beta)\log_2(0.85-0.7\beta) \\
&\quad +0.15\log_2(0.15)+0.85\log_2(0.85)
\end{aligned}
$$

[30%]

(c) Since $H(Y|X)$ does not depend on β, $I(Y;X)$ is maximised when $H(Y)$ is maximised. This will be when $P(y=1)=P(y=-1)=0.5$, which is when $\beta=0.5$. This is well known (see Fig. at the end of section 1 of lecture notes) but can be proved by differentiating

$$
\begin{aligned}
\frac{d}{dp}[H(p)\ln 2] &= \frac{d}{dp}[-p\ln(p)-(1-p)\ln(1-p)] \\
&= -\ln(p)-\frac{p}{p}+\ln(1-p)+\frac{1-p}{1-p} \\
&= \ln\left(\frac{1-p}{p}\right)=0
\end{aligned}
$$

which gives $(1-p)/p=1$ or $p=0.5$. So, $0.15+0.7\beta=0.5$, or $\beta=0.5$.

When $\beta=0.5$,

$$
\begin{aligned}
I(Y;X) &= H([0.5\quad 0.5])-H([0.15\quad 0.85]) \\
&= 1+0.15\log_2(0.15)+0.85\log_2(0.85) \\
&= 1-0.4105-0.1993 \\
&= 0.3902 \text{ bits per bit transmitted}
\end{aligned}
$$

which is the capacity of the channel. Hence, to convey 1000 bits error-free, we need to transmit at least $1000/0.3902=2564$ bits over the channel. In practice, when using real codes we would need a few more bits than this, say 2600. [35%]

END OF PAPER

Module 3F1, April 2009 – SIGNALS AND SYSTEMS – Answers

1 (b)(i) $\frac{Y(z)}{U(z)} = \frac{1-z^{-n}}{1-\beta z^{-n}}$.
(b) (ii) For $n = 1$ zero at 1 and pole at β. For $n = 2$ zeros at ± 1 and poles at $\pm\sqrt{\beta}$.

2 (a) (i) $\frac{1-\beta}{\alpha} < k < \frac{1+\beta}{\alpha}$.
(ii) $\frac{\beta-1}{\beta-1+k\alpha}$.
(ii) $\beta = 1, 0 < k < 2$.
(b) (i) $S_y(\omega) = |H(\omega)|^2$.

3 (c) $\Phi_Z(u) = \frac{1}{(1+u^2x_0^2)(1+u^2y_0^2)}$

4 (b) $I(Y;X) = -(0.15+0.7\beta)\log_2(0.15+0.7\beta) - (0.85-0.7\beta)\log_2(0.85-0.7\beta) + 0.15\log_2(0.15) + 0.85\log_2(0.85)$.
(c) $\beta = 0.5$, capacity of the channel $= 0.3902$ bits per bit transmitted.