3F2 Systems and Control: 2009 Solutions

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1. (a) The state-transition matrix gives the free response of the state from an initial condition (ie when there is no input signal:

$$x(t_1) = e^{A(t_1-t_0)}x(t_0) \tag{1}$$

(OK to have t instead of t_1 and 0 instead of t_0.)

It also appears in the convolution integral for the response of the state to an input signal:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{2}$$

(b) This can be verified by multiplying both sides by $(sI - A)$:

$$LHS = I \tag{3}$$

$$RHS = (sI-A)\left(I\frac{1}{s} + A\frac{1}{s^2} + A^2\frac{1}{s^3} + \cdots\right) \tag{4}$$

$$= I - \frac{A}{s} + \frac{A}{s} - \frac{A^2}{s^2} + \frac{A^2}{s^2} + \cdots \tag{5}$$

$$= I \tag{6}$$

which verifies the claimed equivalence.

Taking the inverse Laplace transform of both sides in the question gives:

$$\mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\} = \mathcal{L}^{-1}\left\{I\frac{1}{s} + A\frac{1}{s^2} + A^2\frac{1}{s^3} + \cdots\right\} \tag{7}$$

$$= I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots \tag{8}$$

$$= e^{At} \quad \text{by definition.} \tag{9}$$

(c) Let x_K be the state vector of the controller, and x_G be the state vector of the plant. Then

$$\dot{x}_K = A_K x_K + B_K e, \qquad u = C_K x_K + D_K e \tag{10}$$

and

$$\dot{x}_G = A_G x_G + B_G u, \qquad y = C_G x_G \tag{11}$$

Substituting for u from (10) in (11) gives

$$\dot{x}_G = A_G x_G + B_G(C_K x_K + D_K e) \tag{12}$$

hence we have, for the series connection:

$$\begin{bmatrix} \dot{x}_K \\ \dot{x}_G \end{bmatrix} = \begin{bmatrix} A_K & 0 \\ B_G C_K & A_G \end{bmatrix} \begin{bmatrix} x_K \\ x_G \end{bmatrix} + \begin{bmatrix} B_K \\ B_G D_K \end{bmatrix} e \tag{13}$$

$$y = \begin{bmatrix} 0 & C_G \end{bmatrix} \begin{bmatrix} x_K \\ x_G \end{bmatrix} \tag{14}$$

Substituting in from the given realisations, and using $(A_{GK}, B_{GK}, C_{GK}, D_{GK})$ to denote the matrices appearing in (13) and (14) gives:

$$A_{GK} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & -2 \end{bmatrix}, \quad B_{GK} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{15}$$

$$C_{GK} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad D_{GK} = 0 \tag{16}$$

Note: Different ordering of the state variables in the combined state vector will give different answers.

(d) i. Let x denote the state vector $[x_K^T, x_G^T]^T$ of the series connection. Substituting $e = r - y = r - C_{GK}x$ into (13) gives:

$$\dot{x} = A_{GK}x + B_{GK}(r - C_{GK}x) = (A_{GK} - B_{GK}C_{GK})x + B_{GK}r \tag{17}$$

$$y = C_{GK}x \tag{18}$$

where

$$A_{cl} = A_{GK} - B_{GK}C_{GK} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \tag{19}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 1 \\ 0 & -1 & -2 \end{bmatrix} \tag{20}$$

ii. The closed-loop poles lie at the roots of

$$\det(sI - A_{cl}) = \det \begin{bmatrix} s & 1 & 0 \\ -1 & s+1 & -1 \\ 0 & 1 & s+2 \end{bmatrix} \tag{21}$$

$$= s[(s+1)(s+2)+1] + 1(s+2) \tag{22}$$

$$= s^3 + 3s^2 + 4s + 2 \tag{23}$$

Verify that -1 is a root: $(-1)^3 + 3(-1)^2 + 4(-1) + 2 = 0$.
So take $s+1$ out as a factor:

$$s^3 + 3s^2 + 4s + 2 = (s+1)(s^2 + 2s + 2) \tag{24}$$

and the roots of the quadratic factor are $(-2 \pm \sqrt{4-8})/2 = -1 \pm j$.

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2. (a) A damping factor no smaller than $1/\sqrt{2}$ will be ensured by each of the closed-loop poles having damping factor no smaller than $1/\sqrt{2}$. That is, all the closed-loop poles should lie between the lines radiating out from the origin into the left half-plane making angles $\pm 45°$ with the negative real axis.

(b) Check the *angle criterion* for the point $z = -2 + 2j$. With the angle α defined as in Fig. 1 we need $2\alpha = \pi/2$, since $\arg(z+2) = \pi/2$, and there are two poles at -4. We see by inspection that this condition is satisfied.

Alternative: Substitute $-2 + 2j$ into $1 + kL(s) = 0$ and show that a (real, positive) k exists which solves this equation. If k is found at the same time, that also solves part (d).

Another alternative: Just check that $L(-2 \pm 2j)$ is real and negative.

(c) To sketch the root-locus diagram go through the standard steps (assuming $k > 0$):

i. The root-locus lies on the real axis between -2 and -4, and to the left of -4 (to the left of an odd number of poles and zeros).

ii. There are 3 loci going off to ∞, since there are 3 poles and no zeros. The asymptotes make angles of $2n\pi/3$ with the negative real axis ($n = 0, 1, 2$).

iii. The asymptotes intersect the real axis at the 'centre of gravity' of the poles and zeros, namely at $(-2 - 4 - 4)/3 = -10/3$.

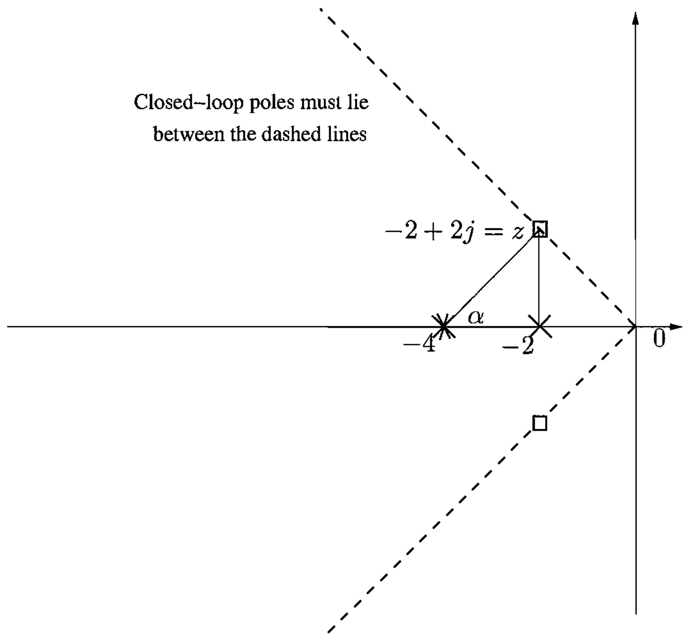

Figure 1: Figure for checking angle condition for 2(b) and for calculating gains for 2(d).

iv. There is a breakaway point somewhere between -2 and -4. Not really necessary to find it accurately, but can be done if desired: Breakaway points are at solutions of

$$\frac{d}{ds}\left\{\frac{8}{(s+2)(s+4)^2}\right\} = 0 \tag{25}$$

$$\frac{-8[(s+4)^2+2(s+2)(s+4)]}{[(s+2)(s+4)^2]^2} = 0 \tag{26}$$

$$(s+4)(3s+8) = 0 \tag{27}$$

so the breakaway points are at -4 and $-8/3$. But there is only one breakaway point needed on the diagram, and it clearly is not at -4. So it is the one at $-8/3$ (the one at -4 being the breakaway point on the locus for $k<0$).

v. In addition, we already know from (b) that the loci pass through $-2 \pm 2j$.

Thus the locus is as shown in Fig. 2.

(d) The gain required to put a closed-loop pole at $z=-2+2j$ is obtained as (using magnitude condition; see Fig. 1):

$$8k = |z+2| \times |z+4|^2 = 2 \times (2\sqrt{2})^2 = 16 \tag{28}$$

The gain we get from this calculation is the required value of $8k$, since the numerator of the transfer function is 8 (rather than monic). So we need $\boxed{k=2}$.

(e) To obtain steady-state error no bigger than 5% requires a zero-frequency loop-gain (ie $8k/(2\times 4^2)$) of at least 20, ie $k>80$ approximately. But we saw in (d) that $8k/32 = 16/32 = 1/2$ puts a pair of closed-loop poles on the boundary of the specification defined in (a), and we can see from (c) that making the gain any larger will take these poles outside this boundary. So it is not possible to meet the requirements of both (a) and (e) with proportional control.

Note: Most candidates did not know that keeping steady-state error below 5% requires zero-frequency loop-gain bigger than 20, approx. Many worked this out in detail, using $|S(0)| = |1/1+kL(0)| < 0.05$. The accurate value obtained this way is $k>76$.

(f) A frequency-dependent controller is needed. At least one of the form

$$k\frac{s+a}{s+b} \tag{29}$$

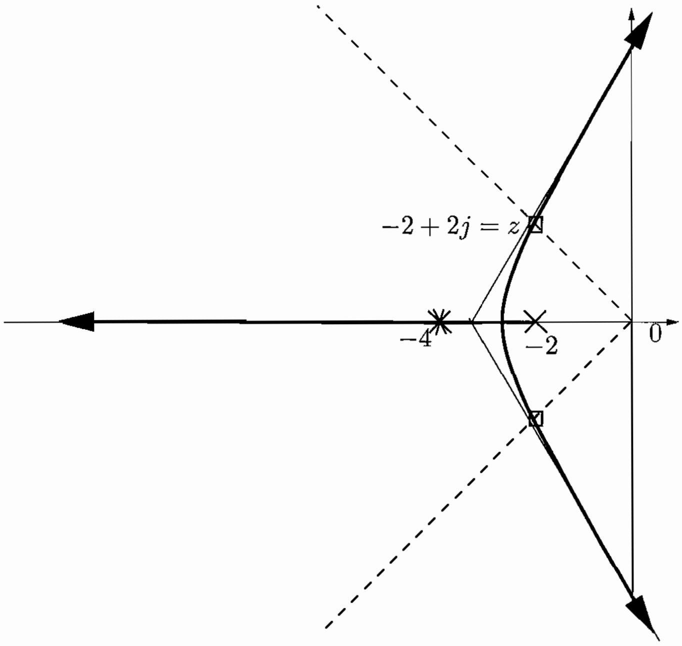

Figure 2: Root-locus diagram for question 2(c), showing asymptotes and the boundary of closed-loop pole locations to satisfy 2(a).

possibly with $b = 0$ to get integral action and thus ensure zero steady-state error, and a sufficiently negative to get enough damping. But this may not be possible, so a lead-lag controller of the form

$$k\frac{(s+a)(s+c)}{(s+b)(s+d)} \tag{30}$$

may be needed, the lag raising gains at low frequencies and the lead giving phase advance at higher frequencies for stabilisation.

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3. (a) The input should clearly be taken to be the voltage u. Equations (1)–(2) are then easily rewritten in the standard state-space form as:

$$\frac{dx}{dt} = Ax + Bu = \begin{bmatrix} 0 & 1 & 0 \\ \frac{S}{M} & 0 & -\frac{G}{M} \\ 0 & \frac{S}{G} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u \tag{31}$$

taking the state vector $x = [\gamma, \dot{\gamma}, I]^T$ as directed. Since we are told that only the current I is measured, the output equation is

$$I = Cx + Du = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x + 0u \tag{32}$$

(b) To test stability we need to evaluate the characteristic polynomial of A (where we use I_3 to denote the identity matrix, to avoid confusion with the current I):

$$\det(sI_3 - A) = \det \begin{bmatrix} s & -1 & 0 \\ -\frac{S}{M} & s & \frac{G}{M} \\ 0 & -\frac{S}{G} & s+\frac{R}{L} \end{bmatrix} \tag{33}$$

$$= s\begin{vmatrix} s & \frac{G}{M} \\ -\frac{S}{G} & s+\frac{R}{L} \end{vmatrix} + \frac{S}{M}\begin{vmatrix} -1 & 0 \\ -\frac{S}{G} & s+\frac{R}{L} \end{vmatrix} \tag{34}$$

$$= s\left[s\left(s+\frac{R}{L}\right)+\frac{S}{M}\right] - \frac{S}{M}\left(s+\frac{R}{L}\right) \tag{35}$$

$$= s^3 + \frac{R}{L}s^2 - \frac{SR}{ML} \tag{36}$$

The coefficient of s is 0 and the coefficient of s^0 is negative. Thus, by the Routh-Hurwitz criterion, not all the roots of this have negative real parts, and so the system is not asymptotically stable.
Alternative: The determinant is the product of the eigenvalues. Now $\det(A) = RS/LM > 0$, so there must be 0 or two eigenvalues with negative real parts, ie at least one eigenvalue with real part zero or positive.

(c) To check observability we have to check whether $\text{rank}(Q) = 3$, where Q is the observability matrix:

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{S}{G} & -\frac{R}{L} \\ \frac{S^2}{GM} & -\frac{RS}{GL} & -\frac{S}{M} + \left(\frac{R}{L}\right)^2 \end{bmatrix} \tag{37}$$

By inspection, each column is linearly independent of the others (since we are told that $S > 0$), so the rank is 3, and the system is observable. *(Alternatively, the determinant of Q could be evaluated:* $\det Q = -S^3/(G^2M)$*.)*
Note: Quite a few candidates evaluated A^2, then multiplied it by C, instead of computing CA^2 as $(CA)A$, which is much quicker. This was even more significant in question 4, where A^2B and A^3B are required.

(d) The standard block-diagram for an observer/state-feedback combination should appear here, either with generic A, B, C values, or specific to this application. See Fig. 3 for a suitable example.

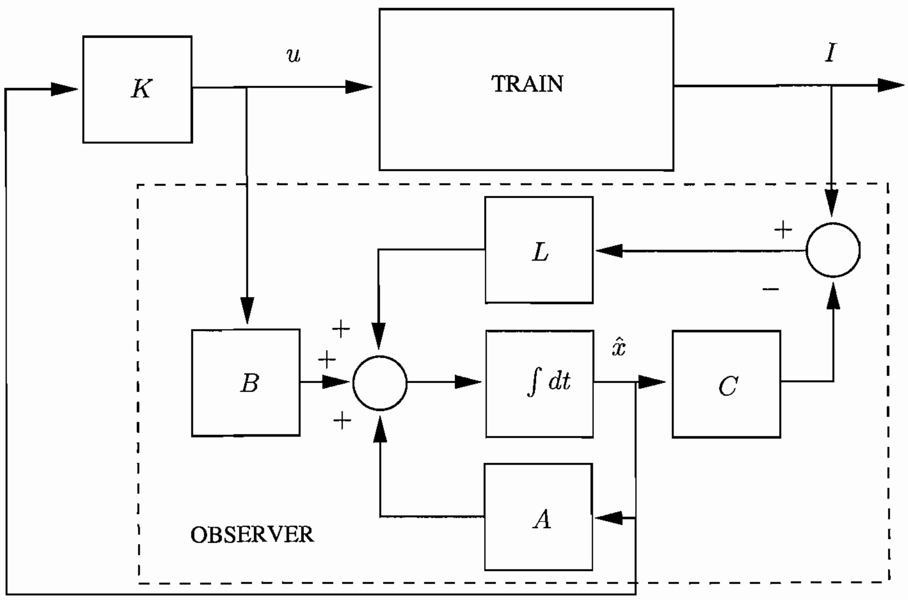

Figure 3: Observer and state-feedback combination for controlling the train.

(e) Although measurement of I alone is in principle enough to estimate all the state variables, in practice it would be desirable to measure more of the states, even though it is probably harder. γ is likely to be easier to measure than its derivative. Reasons for measuring γ as well as I include:

- Measurements are always noisy. Making more measurements allows the signal-to-noise ratio to be increased, and hence to get better estimates of the state vector. (*Not really expected from candidates*: The observer gain can be chosen to give a Kalman filter, giving optimal estimates of the states.)
- Knowledge of the air-gap is safety-critical, hence it is desirable to have a back-up measurement in case either the current measurement, or the observer itself, stops working.
- The linearised equations only hold near equilibrium conditions. The train will have to be accelerated and decelerated, with switchings from magnetic to conventional suspension at times. At such times accurate knowledge of the air-gap is likely to be required, but the observer theory may no longer hold.

Further reasonable reasons can be given credit.

4. (a) A linear system is controllable if there exists an input trajectory $u(\tau)$, $t_0 \leq \tau \leq t_1$, which transfers the state between two arbitrary states $x(t_0)$ and $x(t_1)$ in an arbitrary time interval such that $t_0 < t_1$. *(Alternative equivalent definitions are equally good.)*

Let n be the dimension of the state vector x. Construct the controllability matrix:

$$P = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \tag{38}$$

Then the system is controllable if and only if

$$\text{rank}(P) = n \tag{39}$$

(b) The reachable states are defined as the set of states that can be reached from $x(0) = 0$ by applying some control input $u(t)$ for $0 \leq t \leq t_f$, for some $t_f > 0$.

The reachable states are those states that lie in the range space of P, ie:

$$\text{reachable states} = \{x : x = Pw\} \text{ for some vector } w.$$

(c) In this case the controllability matrix is:

$$P = \begin{bmatrix} 0 & 0 & \frac{1}{m} & 0 & \cdots \\ \frac{1}{m} & 0 & 0 & \frac{2\omega}{m} & \cdots \\ 0 & 0 & 0 & \frac{1}{m} & \cdots \\ 0 & \frac{1}{m} & -\frac{2\omega}{m} & 0 & \cdots \end{bmatrix} \tag{40}$$

There is no need to work out further columns of P, since it is clear that these four columns are linearly independent of each other (since $m < \infty$). Consequently we see that

$$\text{rank}(P) = 4 \tag{41}$$

so the system is controllable.

(d) If the thrusters in the y direction fail and only those in the z direction remain active, then the system is reduced to having only one input (F_z) and the B matrix is replaced by its second column

$$b_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m} \end{bmatrix} \tag{42}$$

Now to test controllability we form the matrix

$$P_2 = [b_2, Ab_2, A^2b_2, A^3b_2] \tag{43}$$

$$= \begin{bmatrix} 0 & 0 & \frac{2\omega}{m} & 0 \\ 0 & \frac{2\omega}{m} & 0 & -\frac{2\omega^3}{m} \\ 0 & \frac{1}{m} & 0 & -\frac{\omega^2}{m} \\ \frac{1}{m} & 0 & -\frac{\omega^2}{m} & 0 \end{bmatrix} \tag{44}$$

Now it can be seen that the 4th column of P_2 is the same as the 2nd column, multiplied by $-\omega^2$, so that the 4 columns are not linearly independent of each other, and hence

$$\text{rank}(P_2) < 4 \tag{45}$$

so the system is no longer controllable.

In this case the reachable states are those states that are in the range space of P_2, namely any state that is a linear combination of the columns of P_2:

$$\text{reachable states} = \left\{ x : x = \alpha \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m} \end{bmatrix} + \beta \begin{bmatrix} 0 \\ \frac{2\omega}{m} \\ \frac{1}{m} \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} \frac{2\omega}{m} \\ 0 \\ 0 \\ -\frac{\omega^2}{m} \end{bmatrix} \right\} \tag{46}$$

3F2 Systems and Control: 2009 Numerical answers

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1. (d)(ii): $-1 \pm j$
2. (d): $k = 2$
3. —
4. —