ENGINEERING TRIPOS PART IIA

Thursday 23 April 2009 9.00 to 12.00

Module 3A1

FLUID MECHANICS I

*Answer not more than **five** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments: 3A1 Data Sheet for Applications to External Flows (2 pages), Boundary Layer Theory Data Card (1 page) and Incompressible Flow Data Card (2 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 Two vortices are located at the points $(x,y) = (0,a)$ and $(x,y) = (0,-a)$ in a uniform flow of velocity U (see Fig. 1). The upper vortex has a circulation of $-\Gamma$ and the lower one has a circulation of Γ.

(a) Write down the complex potential for this flow. [10%]

(b) Find the positions of the stagnation points. [30%]

(c) Consider, now, the case where the distance between the vortices is reduced and their circulation is increased such that as $a \to 0$, $\Gamma a = \text{constant} = C$. Write down the complex potential in the limit as $a \to 0$. Note: $\ln(1+x) \to x$ as $x \to 0$. [40%]

(d) Using the complex potential from (c):

(i) find the stagnation points for this flow;

(ii) describe the size and shape of the body that this may be used to model;

(iii) sketch the flow pattern. [20%]

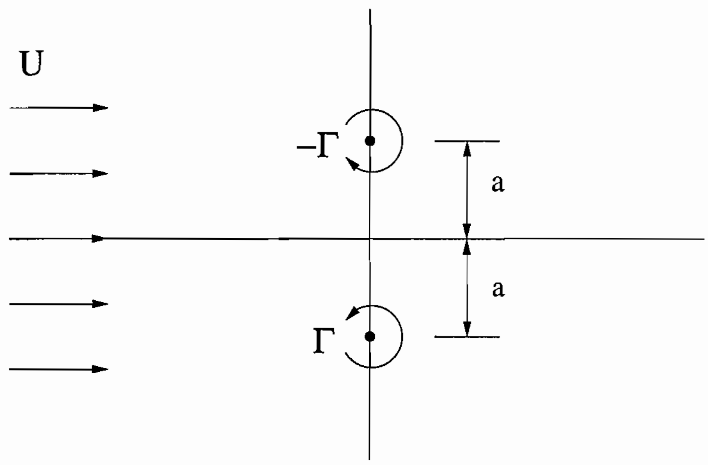

Fig. 1

2 Flow over a rise is to be modelled using one streamline from a half-Rankine body flow. The streamline chosen to best model the geometry is given by the stream function $\psi = 1$. Here the stream function constant is determined by the condition $\psi = 0$ on the positive x-axis.

(a) Write down the complex potential for this flow as shown in Fig. 2. [20%]

(b) Find the change in height of the $\psi = 1$ streamline from far upstream to far downstream (from $x \rightarrow -\infty$ to $x \rightarrow \infty$), in terms of the source strength m and the uniform velocity U. [30%]

(c) Find the difference in pressure between a point located far upstream and a point located on the streamline corresponding to $x = 0$. Express this as a non-dimensional pressure coefficient $\Delta p/(\frac{1}{2}\rho U^2)$, where Δp is the pressure difference of interest. [50%]

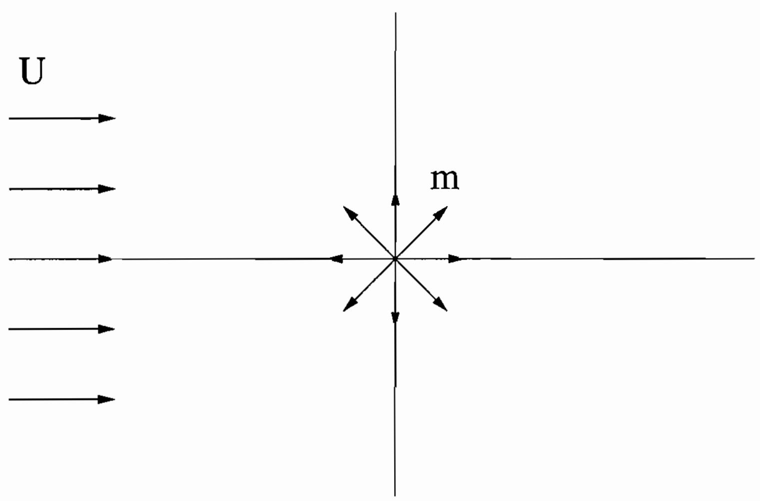

Fig. 2

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3 Consider the inviscid flow of an *incompressible* liquid outside a spherical bubble of radius $R(t)$, where R is a function of time t. The inner motion in the bubble is neglected; a uniform pressure p_b is assumed inside it.

(a) Outside the bubble, the motion is taken to be radial, and the velocity is spherically symmetric. Show that the radial velocity u in the liquid (assumed to depend only on the radial coordinate r and time t) is

$$u(r,t) = \frac{R^2}{r^2}\dot{R}, \quad \text{where } \dot{R} = \frac{dR}{dt}.$$

Calculate the velocity potential

$$\phi = -\int_r^\infty u dr.$$

[20%]

(b) Find the differential equation governing the radius of the bubble in terms of p_b, the pressure at infinity p_∞ and the liquid density ρ. You can use the unsteady form of Bernoulli's equation for irrotational flow:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}u^2 + p/\rho = \text{constant}.$$

[40%]

(c) The frequency of the oscillation depends on the radius of the bubble. The gas inside the bubble behaves isentropically, that is $p_b = KR^{-3\gamma}$ where K is a constant and γ the ratio of the specific heats c_p/c_v. Suppose that $R = a$ is the equilibrium radius of the bubble so that $Ka^{-3\gamma} = p_\infty$.

(i) Assume a small oscillation: $R(t) = a + x(t)$, where $|x/a| \ll 1$. Substitute into the equation obtained in (b). By neglecting all nonlinear terms ($x\ddot{x}$, $\dot{x}^2$, etc), deduce the governing linear differential equation for x

$$a\frac{d^2x}{dt^2} + \frac{3\gamma p_\infty}{\rho a}x = 0. \qquad (1)$$

Note: $(1+\varepsilon)^{-3\gamma} \approx 1 - 3\gamma\varepsilon$ if $\varepsilon \ll 1$. [30%]

(ii) Comment on the similarity between Eqn. (1) and the spring-mass system. How is the frequency of the oscillation related to the radius of the bubble a? [10%]

4 A turbulent boundary layer develops along a plane wall. The velocity profile for all streamwise locations is well-approximated from a region not too close to the wall to the edge of the layer ($y/\delta = 1$) by

$$\frac{U}{U_\tau} = \frac{1}{\kappa}\ln\left(\frac{yU_\tau}{\nu}\right) + C \quad \text{for} \quad yU_\tau/\nu > 100$$

where U is the velocity at a distance y above the wall, $U_\tau = \sqrt{\tau_w/\rho}$ is the wall friction velocity (τ_w is the shear-stress at the wall) and δ is the boundary layer thickness. The von Karman constant may be taken as $\kappa = 0.4$ and C is a constant with the value $C = 5.0$. The kinematic viscosity of the fluid is $\nu = 1.5 \times 10^{-5}$ m^2 s^{-1}.

At the location of interest, the external velocity outside the boundary layer is $U_1 = 36$ m s $^{-1}$. The above logarithmic law is valid at a height of 30 mm above the wall, where the velocity gradient in the y-direction is measured and found to be 100 s^{-1}.

(a) Find the local skin friction coefficient for the boundary layer at this location. [20%]

(b) What is the local Reynolds number ($\delta U_\tau/\nu$) of the flow at this location? [20%]

(c) What is the velocity gradient at the wall ($y \to 0$)? [30%]

(d) The rate of growth of the momentum thickness with streamwise distance is measured at this location to be $d\theta/dx = 0.0111$. What is the pressure-gradient in the streamwise direction? [30%]

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5 A laminar boundary layer profile is given to a crude approximation by

$$\frac{U}{U_1} = \left(\frac{y}{\delta}\right)^2 \quad \text{for} \quad y < \delta\ ;$$

$$U = U_1 \quad \text{for} \quad y \geq \delta$$

where U is the velocity a distance y above the wall, U_1 is the external flow velocity and δ is the boundary layer thickness.

(a) Find [20%]

(i) the displacement thickness, δ^*;

(ii) the momentum thickness, θ;

(iii) the shape factor, H;

(iv) the local skin friction coefficient, C_f.

(b) Using the results from (a), write down the momentum integral equation for this flow. [20%]

(c) Use the momentum integral equation to relate the external velocity U_1 to the boundary layer thickness δ. [20%]

(d) Close to the wall the external pressure-gradient is balanced by the viscous term in Prandtl's boundary layer equation (the other terms go to zero at the wall). Using this fact and the result from (c), find the variation of U_1 with the streamwise distance x. You may assume that at $x = 0$, $\delta = 0$. [40%]

6 (a) Sketch and describe the horseshoe vortex lumped-parameter model of an unswept lifting wing with an elliptical lift distribution. [20%]

(b) An aircraft is flying horizontally. A long way behind, a second aircraft (also flying horizontally) crosses the flying path of the first at right angles, and a distance h above its wake. The horseshoe vortex model may be assumed to apply.

(i) Derive an expression for the downwash experienced by the second aircraft as a function of distance along its flight path. Ensure that your nomenclature is clearly defined. [20%]

(ii) Assuming that the attitude of the second aircraft remains unchanged, describe the translational accelerations it will undergo due to the downwash. [20%]

(c) Now consider the closer encounter shown in Fig. 3.

(i) Find an expression for the downwash experienced by the second aircraft when it is above the centre-line of the first, as a function of the offset l. [20%]

(ii) For $l = 0$, how would you expect the translational acceleration to compare with that experienced above the centre-line in the downstream case considered in (b)? [10%]

(iii) Also for $l = 0$, would you expect the second aircraft to experience any rotational accelerations? Explain your answer. [10%]

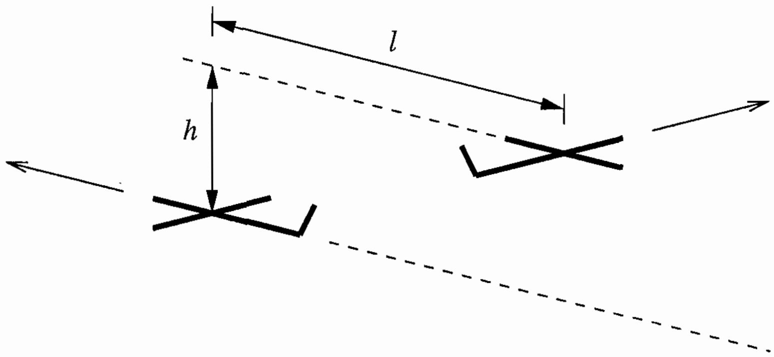

Fig. 3

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7 (a) Figure 4 shows surface pressure coefficient distributions for two aerofoil sections, A and B, at zero incidence. One has thickness-to-chord ratio 6%, and the other 24%. Deduce which is which and identify the other fundamental geometrical difference between the two. Explain your reasoning. Note that Fig.4 (A) shows two coincident lines. [25%]

(b) The incidence of the aerofoils is now increased into their working range. The chord-based Reynolds number is 10^7 (i.e. highly super-critical).

(i) Sketch, to the same scale, the aerofoils' surface pressure coefficient distributions.

(ii) Annotate your sketches to indicate the state of the boundary layer on the upper and lower surfaces. Explain your reasoning.

(iii) Describe the likely stall behaviour of the aerofoils, and discuss how this might be improved without altering the section geometries. [45%]

(c) The Reynolds number is now lowered into the critical range (i.e. around 10^5).

(i) Discuss, with supporting sketches, how the surface pressure distributions and the boundary layer states at working incidence would differ from those in part (b).

(ii) How would you expect the stall behaviour to have altered? Does this imply any change in possible approaches to improving it? [30%]

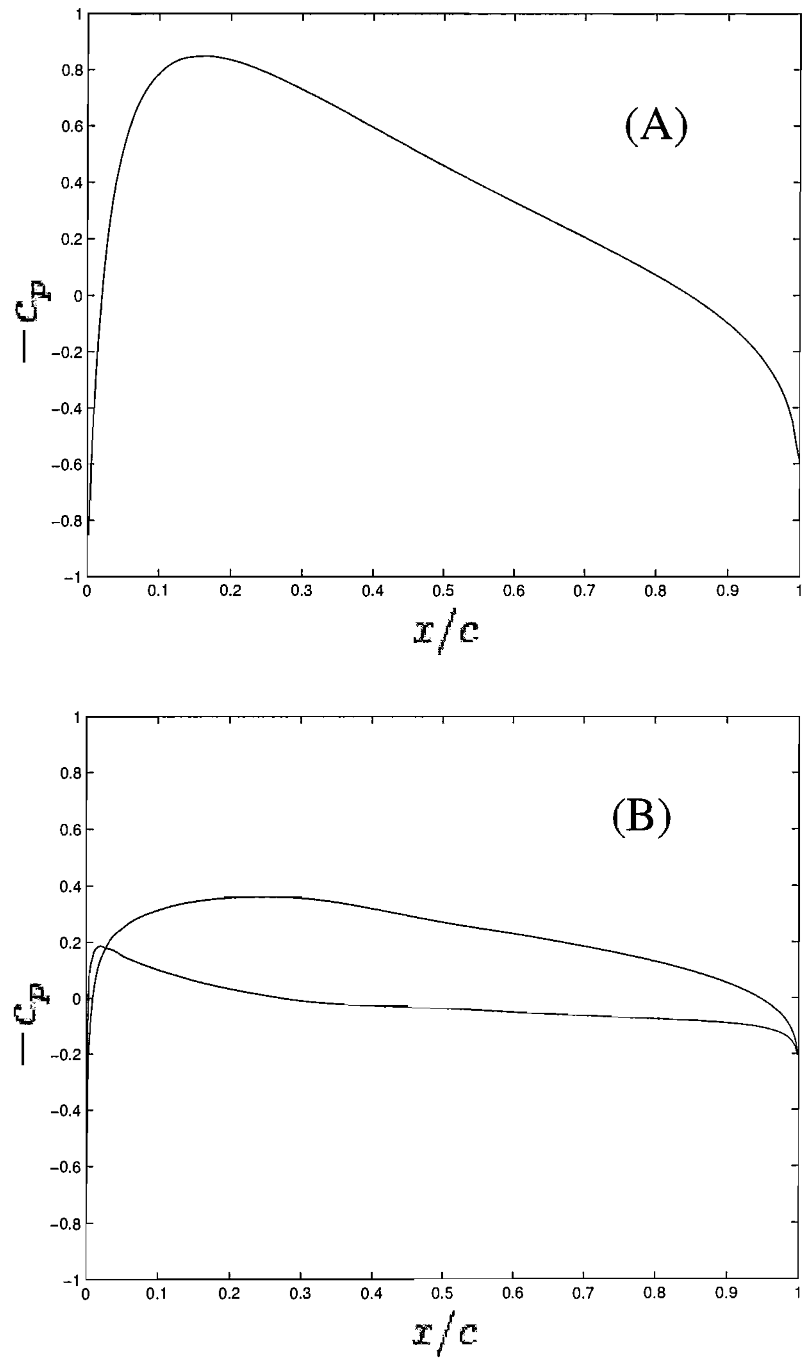

Fig. 4

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8 A low-speed wind-tunnel with a working section of cross-sectional area 1 m^2 is to be constructed in a small laboratory. The available space is 20 m $\times$ 10 m (with about 5 m ceiling height).

(a) Would you choose a closed-loop or an open-loop layout? Explain your choice. Sketch a sensible design for such a tunnel, indicate the key components and, where appropriate, make suggestions for dimensions. [30%]

(b) A motor with a power of 10 kW is available to drive the fan. The design is assumed to be an open-loop layout regardless of your answer in part (a). Ignoring all losses, calculate the maximum flow speed that can be achieved in the working section? What is the power factor of the wind-tunnel? [30%]

(c) In practice, the achievable flow speed will be greatly affected by losses. Suggest where the main contributions to these losses may originate. [10%]

(d) The facility is to be equipped with optical flow diagnostics. Briefly compare the relative merits and drawbacks of PIV vs. LDA. [20%]

(e) For each technique in part (d), give one example of a flow for which the technique would be well-suited and explain why. [10%]

END OF PAPER

3A1 Data Sheet for Applications to External Flows

Aerodynamic Coefficients

For a flow with free-stream density, ρ, velocity U and pressure p_∞:

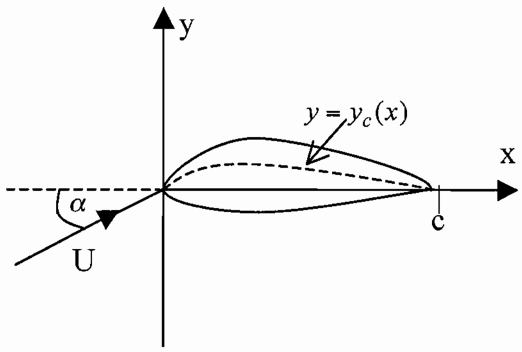
Pressure coefficient: $$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2}$$

Section lift and drag coefficients: $$c_l = \frac{\text{lift}\,(N/m)}{\frac{1}{2}\rho U^2 c},\quad c_d = \frac{\text{drag}\,(N/m)}{\frac{1}{2}\rho U^2 c}$$ (section chord c)

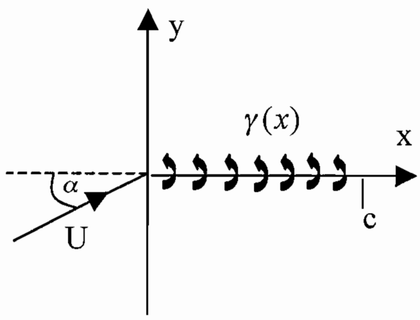
Wing lift and drag coefficients: $$C_L = \frac{\text{lift}\,(N)}{\frac{1}{2}\rho U^2 S},\quad C_D = \frac{\text{drag}\,(N)}{\frac{1}{2}\rho U^2 S}$$ (wing area S)

Thin Aerofoil Theory

Geometry



Approximate representation



Pressure coefficient: $c_p = \pm\gamma / U$

Pitching moment coefficient: $c_m = (\text{moment about } x = 0) / \frac{1}{2}\rho U^2 c^2$

Coordinate transformation: $x = c(1 + \cos\theta)/2 = c\cos^2(\theta/2)$

Incidence solution: $$\gamma = -2U\alpha\frac{1-\cos\theta}{\sin\theta},\quad c_l = 2\pi\alpha,\quad c_m = c_l/4$$

Camber solution: $$\gamma = -U\left[g_0\frac{1-\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} g_n \sin n\theta\right],\ \text{where}$$

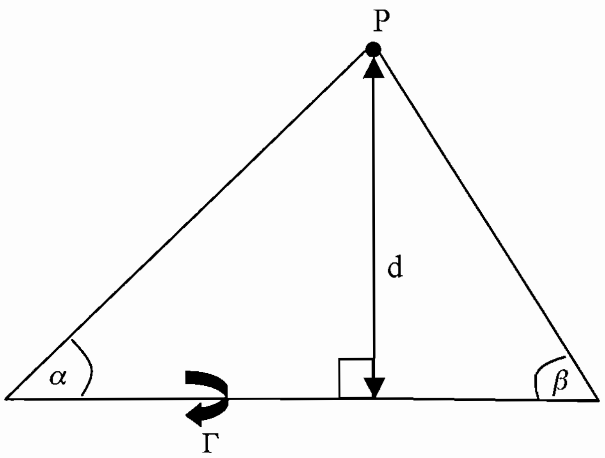
$$g_0 = \frac{1}{\pi}\int_0^\pi\left(-2\frac{dy_c}{dx}\right)d\theta,\quad g_n = \frac{2}{\pi}\int_0^\pi\left(-2\frac{dy_c}{dx}\right)\cos n\theta d\theta$$

$$c_l = \pi\left(g_0 + \frac{g_1}{2}\right),\quad c_m = \frac{\pi}{4}\left(g_0 + g_1 + \frac{g_2}{2}\right) = \frac{c_l}{4} + \frac{\pi}{8}(g_1 + g_2)$$

Glauert Integral

$$\int_{o}^{\pi} \frac{\cos n\phi}{\cos\phi - \cos\theta} d\phi = \pi \frac{\sin n\theta}{\sin\theta}$$

Line Vortices



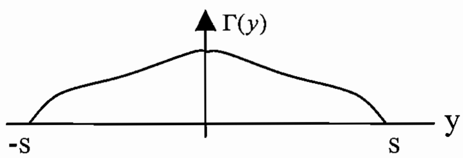
A straight element of circulation Γ induces a velocity at P of

$$\frac{\Gamma}{4\pi d}(\cos\alpha + \cos\beta)$$

perpendicular to the plane containing P and the element.

Lifting-Line Theory

Spanwise circulation distribution:



Aspect ratio: $A_R = 4s^2 / S$

Wing lift: $L = \rho U \int_{-s}^{s} \Gamma(y) dy$

Downwash angle: $\alpha_d(y) = \frac{1}{4\pi U} \int_{-s}^{s} \frac{d\Gamma(\eta)/d\eta}{y - \eta} d\eta$

Induced drag: $D_i = \rho U \int_{-s}^{s} \Gamma(y) \alpha_d(y) dy$

Fourier series for circulation: $\Gamma(y) = Us \sum_{n \text{ odd}} G_n \sin n\theta$, with $y = -s\cos\theta$

Relation between lift and induced drag:

$$C_{Di} = (1+\delta)\frac{C_L^2}{\pi A_R}, \text{ where } \delta = 3\left(\frac{G_3}{G_1}\right)^2 + 5\left(\frac{G_5}{G_1}\right)^2 + \ldots$$

Module 3A1
Boundary Layer Theory Data Card

Displacement thickness;

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_1}\right) dy$$

Momentum thickness;

$$\theta = \int_0^\infty \frac{(U_1 - u)u}{U_1^2} dy = \int_0^\infty \left(1 - \frac{u}{U_1}\right) \frac{u}{U_1} dy$$

Energy thickness;

$$\delta_E = \int_0^\infty \frac{(U_1^2 - u^2)u}{U_1^3} dy = \int_0^\infty \left(1 - \left(\frac{u}{U_1}\right)^2\right) \frac{u}{U_1} dy$$

$$H = \frac{\delta^*}{\theta}$$

Prandtl's boundary layer equations (laminar flow);

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{dp_1}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}$$

von Karman momentum integral equation;

$$\frac{d\theta}{dx} + \frac{H+2}{U_1}\theta\frac{dU_1}{dx} = \frac{\tau_o}{\rho U_1^2} = \frac{C_f'}{2}$$

Boundary layer equations for turbulent flow;

$$\begin{aligned} \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} &= \frac{-1}{\rho}\frac{d\overline{p}}{dx} - \frac{\partial \overline{u'v'}}{\partial y} + \nu\frac{\partial^2 \overline{u}}{\partial y^2} \\ \frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} &= 0 \end{aligned}$$

INCOMPRESSIBLE FLOW DATA CARD

Continuity equation $\nabla \cdot \boldsymbol{u} = 0$

Momentum equation (inviscid) $\rho \dfrac{D\boldsymbol{u}}{Dt} = -\nabla p + \rho \boldsymbol{g}$

D/Dt denotes the material derivative, $\partial/\partial t + \boldsymbol{u} \cdot \nabla$

Vorticity $\boldsymbol{\omega} = \text{curl } \boldsymbol{u}$

Vorticity equation (inviscid) $\dfrac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$

Kelvin's circulation theorem (inviscid) $\dfrac{D\Gamma}{Dt} = 0, \quad \Gamma = \oint \boldsymbol{u} \cdot d\boldsymbol{l} = \int \boldsymbol{\omega} \cdot d\boldsymbol{S}$

For an irrotational flow

velocity potential (ϕ) $\boldsymbol{u} = \nabla \phi$ and $\nabla^2 \phi = 0$

Bernoulli's equation for inviscid flow,

$\dfrac{p}{\rho} + \frac{1}{2} V^2 + gz + \dfrac{\partial \phi}{\partial t}$ = constant throughout flow field, $V = |\boldsymbol{u}|$.

TWO-DIMENSIONAL FLOW

Streamfunction (ψ)

$$u = \frac{\partial \psi}{\partial y}, \qquad \upsilon = -\frac{\partial \psi}{\partial x}$$

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad u_\theta = -\frac{\partial \psi}{\partial r}$$

Lift force Lift / unit length $= \rho U(-\Gamma)$

Complex potential $F(z)$ for irrotational flows, with $z = x + iy$, $F(z) = \phi + i\psi$ and $\dfrac{dF}{dz} = u - i\upsilon$

Examples of complex potentials

(i) uniform flow in x direction, $F(z) = Uz$

(ii) source at z_0, $F(z) = \dfrac{m}{2\pi} \ln(z - z_0)$

(iii) doublet at z_0, with axis in x direction, $F(z) = \dfrac{\mu}{2\pi(z - z_0)}$

(iv) anticlockwise vortex at z_0, $F(z) = -\dfrac{i\Gamma}{2\pi} \ln(z - z_0)$

TWO-DIMENSIONAL FLOW

Summary of simple 2 - D flow fields

	ϕ	ψ	circulation	$\boldsymbol{u}$
Uniform flow (towards $+x$)	Ux	Uy	0	$u = U,\ v = 0$
Source at origin	$\frac{m}{2\pi}\ln r$	$\frac{m}{2\pi}\theta$	0	$u_r = \frac{m}{2\pi r},\ u_\theta = 0$
Doublet at origin θ is angle from doublet axis	$\frac{\mu\cos\theta}{2\pi r}$	$-\frac{\mu\sin\theta}{2\pi r}$	0	$u_r = -\frac{\mu\cos\theta}{2\pi r^2},\ u_\theta = -\frac{\mu\sin\theta}{2\pi r^2}$
Anticlockwise vortex at origin	$\frac{\Gamma}{2\pi}\theta$	$-\frac{\Gamma}{2\pi}\ln r$	Γ around origin	$u_r = 0,\ u_\theta = \frac{\Gamma}{2\pi r}$

THREE-DIMENSIONAL FLOW

Summary of simple 3 - D flow fields

	ϕ	$\boldsymbol{u}$
Source at origin	$-\frac{m}{4\pi r}$	$u_r = \frac{m}{4\pi r^2},\quad u_\theta = 0,\quad u_\phi = 0$
Doublet at origin θ is angle from doublet axis	$\frac{\mu\cos\theta}{4\pi r^2}$	$u_r = -\frac{\mu\cos\theta}{2\pi r^3},\quad u_\theta = -\frac{\mu\sin\theta}{4\pi r^3},\quad u_\phi = 0$

November 2007