ENGINEERING TRIPOS PART IIA

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5 May 2009 9.00 to 10.30

Module 3C5

DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
3C5 Dynamics and 3C6 Vibration datasheet (6 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) Four identical solid spheres, each of mass m and radius a, are assembled to form a tetrahedral ornament as shown in Fig. 1(a). Each sphere is in contact with each of the other spheres. Find the principal moments of inertia of the ornament at its centre of mass. [40%]

(b) Four uniform square plates, each of mass m and side length $2a$ are fixed on their edges to a light shaft of length $8a$ as shown in Fig. 1(b). The centre of mass of the assembly is at O and axes x, y and z are aligned as shown, with z along the shaft. The plates are contained in the x-z and y-z planes. For the assembly, find at O:

(i) the moment of inertia I_{zz}, [20%]

(ii) the product of inertia I_{xy}, [20%]

(iii) the product of inertia I_{yz}. [20%]

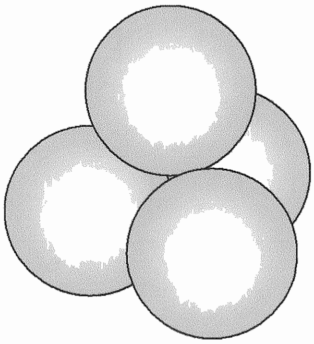

Fig. 1(a)

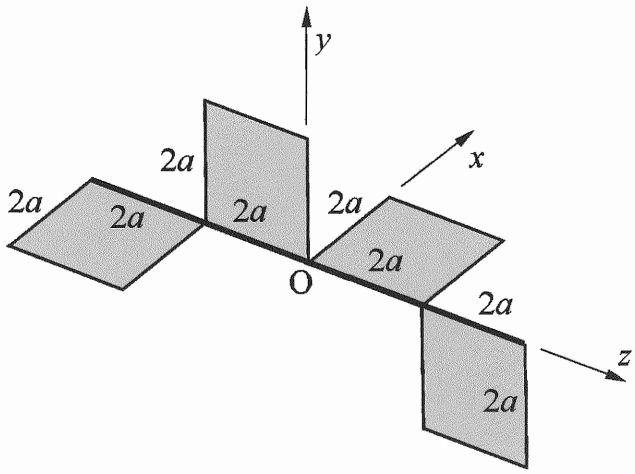

Fig. 1(b)

2 A rotor is fixed to a rigid support through a low-friction spherical bearing at O as shown in Fig. 2. The moments of inertia of the rotor are A, A, C. The centre of mass of the rotor is a distance a from O. A unit vector $\mathbf{k}$ is aligned with the spin axis and the unit vector $\mathbf{K}$ is vertical. The angle of inclination (between $\mathbf{k}$ and $\mathbf{K}$) is θ. The rotor is spinning at a constant fast rate $\omega\mathbf{k}$ and under the action of gravity the rotor precesses slowly at a rate $\Omega\mathbf{K}$.

(a) Show that the precession rate is independent of θ. [50%]

(b) Friction at O, represented by a small constant couple $Q\mathbf{K}$, now causes the rotor to descend.

(i) What is the additional rate of change of moment of momentum of the rotor? [20%]

(ii) When $\theta = \pi/2$ what is the rate of increase of θ? [30%]

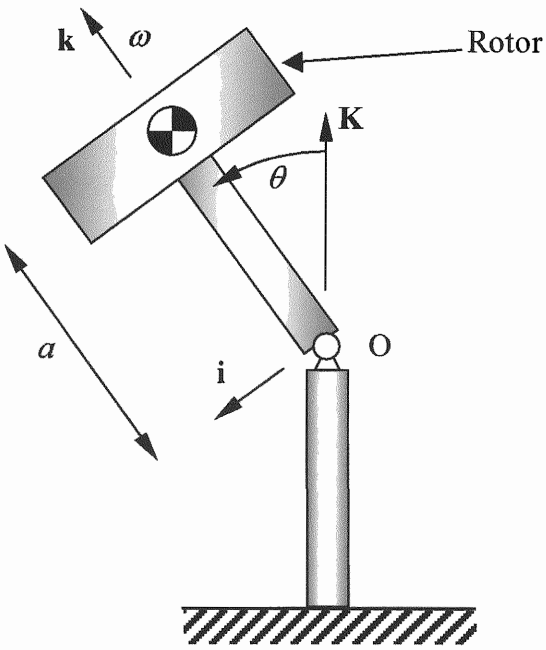

Fig. 2

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3 A bicycle wheel of radius a, mass m and moments of inertia A, A, C, is rolling steadily without slip on a flat horizontal surface. The forward speed of the wheel is V and the point of contact with the surface P traces out a large circle of radius R on the surface. The wheel is inclined to the vertical by a small constant angle α .

(a) On a free-body diagram of the wheel show all forces and hence deduce the couple acting on the wheel. [20%]

(b) Find the radius R of the path for a given speed V and angle α . For the case $C = 2A = ma^2$ show that $R = 2V^2 /(g\alpha)$. [80%]

4 (a) Define the term "generalised coordinate" within the context of Lagrange's equation. Give examples of generalised coordinates for both discrete and continuous dynamic systems. [15%]

(b) Define the term "generalised force", and explain how the generalised force Q_i associated with a particular generalised coordinate q_i can be found. [15%]

(c) The displacement $u(x,t)$ of a simply supported beam is approximated in the form

$$u(x,t) = q_1(t)\sin(\pi x / L) + q_2(t)\sin(2\pi x / L),$$

where $q_1(t)$ and $q_2(t)$ are the generalised coordinates, L is the length of the beam, and the coordinate x is measured from one end of the beam. A force F is applied to the beam at location $x = x_0$. Derive expressions for the generalised forces Q_1 and Q_2. [15%]

(d) The kinetic and potential energies of the beam described above are given by

$$T = (mL/4)\{\dot{q}_1^2 + \dot{q}_2^2\}, \qquad V = (\pi^2 EI / 4L)\{q_1^2 + 4q_2^2\},$$

where m is the mass per unit length and EI is the flexural rigidity. If a point mass M is attached to the beam at the location $x = x_0$, derive an expression for the additional kinetic energy, and hence find the equations of motion of the system. Calculate the natural frequencies for the case $x_0 = L/2$. [30%]

(e) The beam displacement is now written in the more general form

$$u(x,t) = \sum_{j=1}^{N} q_j(t)\sin(j\pi x / L).$$

Derive an expression for the jk'th entry of the mass matrix arising from the presence of the mass M at location $x = x_0$. [25%]

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5 A mass m slides inside a cylinder of radius r which rolls without slip on a flat horizontal surface, as shown in Fig. 3. The motion of the mass is planar, and its position is described by the angle θ_1 between the vertical and a line joining the mass to the centre of the cylinder. The rolling motion of the cylinder is described by the rotation θ_2 about its centre. The mass of the cylinder is M, and it is thin-walled, so that the moment of inertia about its centre is Mr^2. A horizontal force F is applied at the centre of the cylinder.

(a) Show that the kinetic energy of the system is given by

$$T = (mr^2/2)\{\dot{\theta}_1^2 + \dot{\theta}_2^2 - 2\dot{\theta}_1\dot{\theta}_2\cos\theta_1\} + Mr^2\dot{\theta}_2^2 .$$ [15%]

(b) By using Lagrange's equation, derive the equations of motion of the system. Do not assume that either θ_1 or θ_2 is small. [20%]

(c) Show that the mass m can be maintained in a steady position $\theta_1 = \psi$ by the application to the centre of the cylinder of a force of magnitude $F = (m + 2M)g\tan\psi$, where g is the gravitational acceleration. [30%]

(d) Show that, for $F=0$, the natural frequencies of small amplitude oscillations about the static equilibrium position are given by

$$\omega_1 = 0, \qquad \omega_2^2 = \left(\frac{g}{r}\right)\left(\frac{m+2M}{2M}\right).$$ [35%]

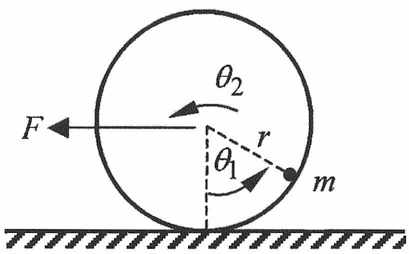

Fig. 3

END OF PAPER

Part IIA Data sheet

Module 3C5 Dynamics

Module 3C6 Vibration

DYNAMICS IN THREE DIMENSIONS

Axes fixed in direction

(a) Linear momentum for a general collection of particles m_i:

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}^{(e)}$$

where $\boldsymbol{p} = M\,\boldsymbol{v}_G$, M is the total mass, $\boldsymbol{v}_G$ is the velocity of the centre of mass and $\boldsymbol{F}^{(e)}$ the total external force applied to the system.

(b) Moment of momentum about a general point P

$$\boldsymbol{Q}^{(e)} = (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \dot{\boldsymbol{p}} + \dot{\boldsymbol{h}}_G$$
$$= \dot{\boldsymbol{h}}_P + \dot{\boldsymbol{r}}_P \times \boldsymbol{p}$$

where $\boldsymbol{Q}^{(e)}$ is the total moment of external forces about P. Here, $\boldsymbol{h}_P$ and $\boldsymbol{h}_G$ are the moments of momentum about P and G respectively, so that for example

$$\boldsymbol{h}_P = \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_P) \times m_i \dot{\boldsymbol{r}}_i$$
$$= \boldsymbol{h}_G + (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \boldsymbol{p}$$

where the summation is over all the mass particles making up the system.

(c) For a rigid body rotating with angular velocity $\boldsymbol{\omega}$ about a fixed point P at the origin of coordinates

$$\boldsymbol{h}_P = \int \boldsymbol{r} \times (\boldsymbol{\omega} \times \mathrm{r})\,dm = \boldsymbol{I}\,\boldsymbol{\omega}$$

where the integral is taken over the volume of the body, and where

$$\boldsymbol{I} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \qquad \boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and
$$A = \int (y^2 + z^2)\,dm \qquad B = \int (z^2 + x^2)\,dm \qquad C = \int (x^2 + y^2)\,dm$$
$$D = \int yz\,dm \qquad E = \int zx\,dm \qquad F = \int xy\,dm$$

where all integrals are taken over the volume of the body.

Axes rotating with angular velocity $\boldsymbol{\Omega}$

Time derivatives of vectors must be replaced by the "rotating frame" form, so that for example

$$\dot{\boldsymbol{p}} + \boldsymbol{\Omega} \times \boldsymbol{p} = \boldsymbol{F}^{(e)}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector $\boldsymbol{r}$ is needed, as in 1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

Euler's dynamic equations (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$A\,\dot{\omega}_1 - (B - C)\,\omega_2\,\omega_3 = Q_1$$
$$B\,\dot{\omega}_2 - (C - A)\,\omega_3\,\omega_1 = Q_2$$
$$C\,\dot{\omega}_3 - (A - B)\,\omega_1\,\omega_2 = Q_3$$

where A, B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

(b) Non-body-fixed reference frame for axisymmetric bodies (the "Gyroscope equations"):

$$A\,\dot{\Omega}_1 - (A\,\Omega_3 - C\,\omega_3)\,\Omega_2 = Q_1$$
$$A\,\dot{\Omega}_2 + (A\,\Omega_3 - C\,\omega_3)\,\Omega_1 = Q_2$$
$$C\,\dot{\omega}_3 = Q_3$$

where A, A and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes such that ω_3 and Q_3 are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\boldsymbol{\Omega} = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$.

Lagrange's equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy, and Q_i are the non-conservative generalised forces.

VIBRATION MODES AND RESPONSE

Discrete systems

1. The forced vibration of an N-degree-of-freedom system with mass matrix M and stiffness matrix K (both symmetric and positive definite) is

$$M\,\ddot{\underline{y}} + K\,\underline{y} = \underline{f}$$

where $\underline{y}$ is the vector of generalised displacements and $\underline{f}$ is the vector of generalised forces.

2. Kinetic energy

$$T = \frac{1}{2}\dot{\underline{y}}^t M \dot{\underline{y}}$$

Potential energy

$$V = \frac{1}{2}\underline{y}^t K \underline{y}$$

3. The natural frequencies ω_n and corresponding mode shape vectors $\underline{u}^{(n)}$ satisfy

$$K\underline{u}^{(n)} = \omega_n^2 M\underline{u}^{(n)}\,.$$

4. Orthogonality and normalisation

$$\underline{u}^{(j)t} M \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

$$\underline{u}^{(j)t} K \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ \omega_n^2, & j = k \end{cases}$$

5. General response

The general response of the system can be written as a sum of modal responses

$$\underline{y}(t) = \sum_{j=1}^{N} q_j(t)\,\underline{u}^{(j)} = U\underline{q}(t)$$

where U is a matrix whose N columns are the normalised eigenvectors $\underline{u}^{(j)}$ and q_j can be thought of as the "quantity" of the jth mode.

Continuous systems

The forced vibration of a continuous system is determined by solving a partial differential equation: see p. 6 for examples.

$$T = \frac{1}{2}\int \dot{u}^2\,dm$$

where the integral is with respect to mass (similar to moments and products of inertia).

See p. 6 for examples.

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see p. 6) and boundary conditions, assuming harmonic time dependence.

$$\int u_j(x)\,u_k(x)\,dm = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

The general response of the system can be written as a sum of modal responses

$$w(x,t) = \sum_j q_j(t)\,u_j(x)$$

where $w(x,t)$ is the displacement and q_j can be thought of as the "quantity" of the jth mode.

6. Modal coordinates $\underline{q}$ satisfy

$$\ddot{\underline{q}} + \left[\text{diag}(\omega_j^2)\right]\underline{q} = \underline{Q}$$

where $\underline{y} = U\underline{q}$ and the modal force vector

$$\underline{Q} = U^t \underline{f} \, .$$

Each modal amplitude $q_j(t)$ satisfies

$$\ddot{q}_j + \omega_j^2 q_j = Q_j$$

where $Q_j = \int f(x,t)\, u_j(x)\, dm$ and $f(x,t)$ is the external applied force distribution.

7. Frequency response function

For input generalised force f_j at frequency ω and measured generalised displacement y_k the transfer function is

$$H(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For force F at frequency ω applied at point x, and displacement w measured at point y, the transfer function is

$$H(x,y,\omega) = \frac{w}{F} = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x,y,\omega) = \frac{w}{F} \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

8. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)} u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with low modal overlap, if the factor $u_n(x)\, u_n(y)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

9. Impulse response

For a unit impulsive generalised force $f_j = \delta(t)$ the measured response y_k is given by

$$g(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin\omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin\omega_n t \, e^{-\omega_n \zeta_n t}$$

for $t \geq 0$ (with small damping).

For a unit impulse applied at $t = 0$ at point x, the response at point y is

$$g(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin\omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(x,y,t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin\omega_n t \, e^{-\omega_n \zeta_n t}$$

for $t \geq 0$ (with small damping).

10. Step response

For a unit step generalised force $f_j = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$ the measured response y_k is given by

$$h(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos\omega_n t\right]$$

for $t \geq 0$ (with no damping), or

$$h(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos\omega_n t \, e^{-\omega_n \zeta_n t}\right]$$

for $t \geq 0$ (with small damping).

For a unit step force applied at $t = 0$ at point x, the response at point y is

$$h(x,y,t) = \sum_{n} \frac{u_n(x)\, u_n(y)}{\omega_n^2} \left[1 - \cos\omega_n t\right]$$

for $t \geq 0$ (with no damping), or

$$h(t) \approx \sum_{n} \frac{u_n(x)\, u_n(y)}{\omega_n^2} \left[1 - \cos\omega_n t \, e^{-\omega_n \zeta_n t}\right]$$

for $t \geq 0$ (with small damping).

Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is $\frac{V}{T} = \frac{\underline{y}^t K \underline{y}}{\underline{y}^t M \underline{y}}$ where $\underline{y}$ is the vector of generalised coordinates, M is the mass matrix and K is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions on p. 6.

If this quantity is evaluated with any vector $\underline{y}$, the result will be

(1) $\geq$ the smallest squared frequency;

(2) $\leq$ the largest squared frequency;

(3) a good approximation to ω_k^2 if $\underline{y}$ is an approximation to $\underline{u}^{(k)}$.

(Formally, $\frac{V}{T}$ is *stationary* near each mode.)

Governing equations for continuous systems

Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $w(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\dfrac{\partial^2 w}{\partial t^2} - P\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}P\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}m\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $f(x,t)$ per unit length.
Polar moment of area is $J = (\pi/2)\left(a^4 - b^4\right)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}GJ\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho J\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$

Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $w(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} - EA\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}EA\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $w(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} + EI\dfrac{\partial^4 w}{\partial x^4} = f(x,t)$	$V = \dfrac{1}{2}EI\int\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.

3C5 Dynamics: Answers to Tripos Paper 2009

1. (a) AAA body with A= $28ma^2/5$.

   (b) $I_{zz} = 16ma^2/3, \quad I_{xy} = 0, \quad I_{yz} = -4ma^2$.

2. (a) $\dot{\phi} = mga/(C\omega)$.

   (b) (i) $\dot{\mathbf{h}} = Q\mathbf{K}$, (ii) $\dot{\theta} = -Q/(C\omega)$.

3. (a) $Q = (mV^2 a/R)\cos\alpha - mga\sin\alpha$.

4. (c) $Q_1 = F\sin(\pi x_0/L), \quad Q_2 = F\sin(2\pi x_0/L)$.

   (d) Additional $T = 0.5[\dot{q}_1 \sin(\pi x_0/L) + \dot{q}_2 \sin(2\pi x_0/L)]^2$,

   $\omega_1^2 = \pi^2 EI/(mL^2 + 2ML), \quad \omega_2^2 = 4\pi^2 EI/(mL^2)$.

   (e) $M_{jk} = M\sin(j\pi x_0/L)\sin(k\pi x_0/L)$.

5. (b) $mr^2\ddot{\theta}_1 - mr^2\ddot{\theta}_2\cos\theta_1 + mgr\sin\theta_1 = 0$,

   $(mr^2 + 2Mr^2)\ddot{\theta}_2 - mr^2\ddot{\theta}_1\cos\theta_1 + mr^2\dot{\theta}_1^2\sin\theta_1 = Fr$.