ENGINEERING TRIPOS PART IIA

Thursday 7 May 2009 9 to 10.30

Module 3C6

VIBRATIONS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
Data Sheet: 3C5 Dynamics and 3C6 Vibration (6 pages)

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) A uniform beam of length L, mass per unit length m and bending stiffness EI can undergo small transverse bending vibration in one plane, with displacement $w(x,t)$. For the case where the beam is clamped to a rigid support at $x = 0$ and is free at $x = L$, find an equation whose roots determine the natural frequencies. Sketch the first three mode shapes. [40%]

(b) The cantilever beam from (a) is now assumed to have a rectangular cross-section with a fixed width b but with a depth h (in the plane of vibration) which varies slightly with x:

$$h = h_0 + \delta h(x), \quad |\delta h(x)| << h_0 \ .$$

Use Rayleigh's principle to derive an approximate expression for the first natural frequency of the variable-depth beam, in terms of the corresponding mode shape $u_1(x)$ and natural frequency ω_1 of a uniform beam. You may assume that the correct expressions for potential and kinetic energy of the variable-depth beam are given by the data sheet expressions, except that the factors I and A should be taken *inside* the relevant integrals. [30%]

(c) Without detailed calculations, explain how the expression from (b) can be used to guide the process of adjusting the shape of a tuning fork in order to bring the lowest natural frequency to a desired value. Given that you can only remove material, not add it, in the course of this adjustment, where should you make adjustments to raise or lower the frequency? [30%]

2 (a) Summarise the key differences between discrete and continuous vibrating systems. Explain why general results proved for discrete systems can be carried over to continuous systems. [20%]

(b) A uniform elastic column of length L, cross-sectional area A, density ρ and Young's modulus E is fixed to a rigid foundation at $x = 0$ and is free at the upper end $x = L$. The column can undergo small axial vibration, with displacement $w(x,t)$. Find the natural frequencies and mode shapes of the column. [30%]

(c) Write down an expression in terms of the modes in part (b) for the response at $x = L/4$ following a unit impulse applied at $x = L$. Deduce a corresponding expression for the velocity response at $x = L/4$. Draw a careful sketch to show the relative contributions of the first three modes to this response. Use a time axis showing one complete cycle of the lowest natural frequency. [40%]

(d) Based on the sketch in part (c), estimate the time when the first peak of velocity occurs at $x = L/4$. Explain what determines this time, and hence give a more accurate value. [10%]

3 Figure 1 shows a light string of length $5L$ with equally spaced masses of m, $2m$, $2m$, m which undergo small displacements $[y_1 \quad y_2 \quad y_3 \quad y_4]^{\mathrm{T}}$. The tension in the string is constant P and the effects of gravity may be ignored.

(a) Write expressions for the potential and kinetic energies of the system. Hence or otherwise show that the stiffness matrix is

$$\frac{P}{L}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

[25%]

(b) Sketch the mode shapes of the system in order of increasing frequency. [25%]

(c) A transverse sinusoidal force f is applied to the left-most mass as shown. Sketch a graph of the magnitude of the steady-state displacement response of y_3, as a function of the frequency of excitation. Use a dB scale and show salient values. [25%]

(d) Explain briefly why the lowest frequency mode must have a shape of the form $[1 \quad \alpha \quad \alpha \quad 1]^{\mathrm{T}}$ where α is a constant. Estimate α and the lowest natural frequency by minimising Rayleigh's quotient using this mode shape. Comment on the accuracy of all natural frequencies and mode shapes found by this calculation. [25%]

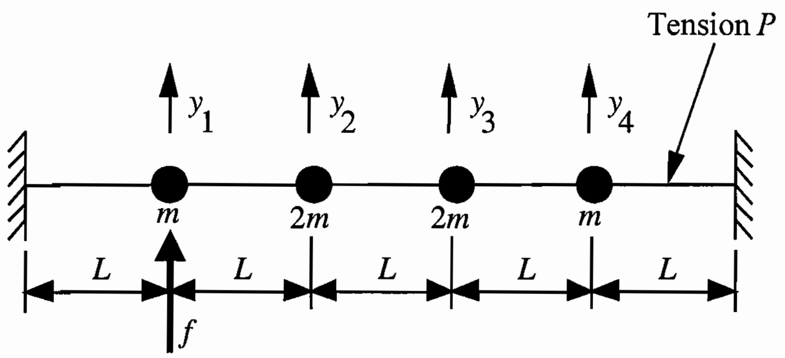

Fig. 1

4 The system shown in Fig. 2 consists of three masses m, $2m$ and m, connected by two light elastic springs, each with stiffness k. The system is constrained to move horizontally. The absolute displacements of the masses are y_1, y_2, y_3. Force input F can be applied to mass 1.

(a) Determine, by inspection or otherwise, the natural frequencies and natural mode shapes of oscillation of the system. [30%]

(b) The left hand mass is forced to vibrate sinusoidally with a displacement amplitude of $y_1 = \Delta$, at a frequency corresponding to the lowest non-zero natural frequency. What is the amplitude and phase of the displacement of the right hand mass y_3? [20%]

(c) The system is initially at rest and in equilibrium. Mass 1 is given a horizontal impulse $F = I\,\delta(t)$ at time $t = 0$. Assuming that k is very large, so that the system is effectively a single rigid body, write down an expression for motion following the impulse. [10%]

(d) The system is initially at rest and in equilibrium. Mass 1 is given a horizontal impulse $F = I\,\delta(t)$ at time $t = 0$. For a general value of k, determine the displacement y_2 of mass 2 at time $t = \sqrt{(m/k)}$. [40%]

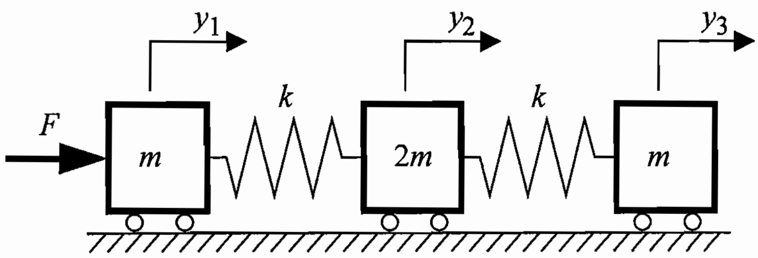

Fig. 2

END OF PAPER

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Part IIA Data sheet

Module 3C5 Dynamics

Module 3C6 Vibration

DYNAMICS IN THREE DIMENSIONS

Axes fixed in direction

(a) Linear momentum for a general collection of particles m_i:

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}^{(e)}$$

where $\boldsymbol{p} = M\,\boldsymbol{v}_G$, M is the total mass, $\boldsymbol{v}_G$ is the velocity of the centre of mass and $\boldsymbol{F}^{(e)}$ the total external force applied to the system.

(b) Moment of momentum about a general point P

$$\boldsymbol{Q}^{(e)} = (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \dot{\boldsymbol{p}} + \dot{\boldsymbol{h}}_G$$
$$= \dot{\boldsymbol{h}}_P + \dot{\boldsymbol{r}}_P \times \boldsymbol{p}$$

where $\boldsymbol{Q}^{(e)}$ is the total moment of external forces about P. Here, $\boldsymbol{h}_P$ and $\boldsymbol{h}_G$ are the moments of momentum about P and G respectively, so that for example

$$\boldsymbol{h}_P = \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_P) \times m_i \dot{\boldsymbol{r}}_i$$
$$= \boldsymbol{h}_G + (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \boldsymbol{p}$$

where the summation is over all the mass particles making up the system.

(c) For a rigid body rotating with angular velocity $\boldsymbol{\omega}$ about a fixed point P at the origin of coordinates

$$\boldsymbol{h}_P = \int \boldsymbol{r} \times (\boldsymbol{\omega} \times \mathbf{r})\,dm = I\,\boldsymbol{\omega}$$

where the integral is taken over the volume of the body, and where

$$I = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \qquad \boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and $A = \int (y^2 + z^2)\,dm \qquad B = \int (z^2 + x^2)\,dm \qquad C = \int (x^2 + y^2)\,dm$

$D = \int yz\,dm \qquad E = \int zx\,dm \qquad F = \int xy\,dm$

where all integrals are taken over the volume of the body.

Axes rotating with angular velocity $\boldsymbol{\Omega}$

Time derivatives of vectors must be replaced by the "rotating frame" form, so that for example

$$\dot{\boldsymbol{p}} + \boldsymbol{\Omega} \times \boldsymbol{p} = \boldsymbol{F}^{(e)}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector $\boldsymbol{r}$ is needed, as in 1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

Euler's dynamic equations (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$A\,\dot{\omega}_1 - (B - C)\,\omega_2\,\omega_3 = Q_1$$
$$B\,\dot{\omega}_2 - (C - A)\,\omega_3\,\omega_1 = Q_2$$
$$C\,\dot{\omega}_3 - (A - B)\,\omega_1\,\omega_2 = Q_3$$

where A, B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

(b) Non-body-fixed reference frame for axisymmetric bodies (the "Gyroscope equations"):

$$A\,\dot{\Omega}_1 - (A\,\Omega_3 - C\,\omega_3)\,\Omega_2 = Q_1$$
$$A\,\dot{\Omega}_2 + (A\,\Omega_3 - C\,\omega_3)\,\Omega_1 = Q_2$$
$$C\,\dot{\omega}_3 = Q_3$$

where A, A and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes such that ω_3 and Q_3 are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\boldsymbol{\Omega} = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$.

Lagrange's equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy, and Q_i are the non-conservative generalised forces.

VIBRATION MODES AND RESPONSE

Discrete systems

1. The forced vibration of an N-degree-of-freedom system with mass matrix M and stiffness matrix K (both symmetric and positive definite) is

$$M\,\ddot{\underline{y}} + K\,\underline{y} = \underline{f}$$

where $\underline{y}$ is the vector of generalised displacements and $\underline{f}$ is the vector of generalised forces.

2. Kinetic energy

$$T = \frac{1}{2}\dot{\underline{y}}^t M \dot{\underline{y}}$$

Potential energy

$$V = \frac{1}{2}\underline{y}^t K \underline{y}$$

3. The natural frequencies ω_n and corresponding mode shape vectors $\underline{u}^{(n)}$ satisfy

$$K\underline{u}^{(n)} = \omega_n{}^2 M\underline{u}^{(n)} .$$

4. Orthogonality and normalisation

$$\underline{u}^{(j)t} M \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

$$\underline{u}^{(j)t} K \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ \omega_n^2, & j = k \end{cases}$$

5. General response

The general response of the system can be written as a sum of modal responses

$$\underline{y}(t) = \sum_{j=1}^{N} q_j(t)\,\underline{u}^{(j)} = U\underline{q}(t)$$

where U is a matrix whose N columns are the normalised eigenvectors $\underline{u}^{(j)}$ and q_j can be thought of as the "quantity" of the jth mode.

Continuous systems

The forced vibration of a continuous system is determined by solving a partial differential equation: see p. 6 for examples.

Kinetic energy:

$$T = \frac{1}{2}\int \dot{u}^2 dm$$

where the integral is with respect to mass (similar to moments and products of inertia).

Potential energy: See p. 6 for examples.

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see p. 6) and boundary conditions, assuming harmonic time dependence.

Orthogonality and normalisation:

$$\int u_j(x)\,u_k(x)\,dm = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

The general response of the system can be written as a sum of modal responses

$$w(x,t) = \sum_j q_j(t)\,u_j(x)$$

where $w(x,t)$ is the displacement and q_j can be thought of as the "quantity" of the jth mode.

6. Modal coordinates $\underline{q}$ satisfy

$$\ddot{\underline{q}} + \left[\mathrm{diag}(\omega_j^2)\right]\underline{q} = \underline{Q}$$

where $\underline{y} = U\underline{q}$ and the modal force vector

$$\underline{Q} = U^t \underline{f} \ .$$

Each modal amplitude $q_j(t)$ satisfies

$$\ddot{q}_j + \omega_j^2 \, q_j = Q_j$$

where $Q_j = \int f(x,t)\, u_j(x)\, dm$ and $f(x,t)$ is the external applied force distribution.

7. Frequency response function

For input generalised force f_j at frequency ω and measured generalised displacement y_k the transfer function is

$$H(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For force F at frequency ω applied at point x, and displacement w measured at point y, the transfer function is

$$H(x,y,\omega) = \frac{w}{F} = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x,y,\omega) = \frac{w}{F} \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

8. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)} u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with low modal overlap, if the factor $u_n(x)\, u_n(y)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

9. Impulse response

For a unit impulsive generalised force $f_j = \delta(t)$ the measured response y_k is given by

$$g(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin\omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin\omega_n t \, e^{-\omega_n\zeta_n t}$$

for $t \geq 0$ (with small damping).

For a unit impulse applied at $t = 0$ at point x, the response at point y is

$$g(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin\omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(x,y,t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin\omega_n t \, e^{-\omega_n\zeta_n t}$$

for $t \geq 0$ (with small damping).

10. Step response

For a unit step generalised force

$$f_j = \begin{cases} 0 & t < 0 \\ 1 & t \ge 0 \end{cases}$$

the measured response y_k is given by

$$h(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos\omega_n t\right]$$

for $t \ge 0$ (with no damping), or

$$h(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos\omega_n t \, e^{-\omega_n \zeta_n t}\right]$$

for $t \ge 0$ (with small damping).

For a unit step force applied at $t = 0$ at point x, the response at point y is

$$h(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2} \left[1 - \cos\omega_n t\right]$$

for $t \ge 0$ (with no damping), or

$$h(t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2} \left[1 - \cos\omega_n t \, e^{-\omega_n \zeta_n t}\right]$$

for $t \ge 0$ (with small damping).

Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is $\dfrac{\bar{V}}{\bar{T}} = \dfrac{\underline{y}^t K \underline{y}}{\underline{y}^t M \underline{y}}$ where $\underline{y}$ is the vector of generalised coordinates, M is the mass matrix and K is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions on p. 6.

If this quantity is evaluated with any vector $\underline{y}$, the result will be

(1) $\ge$ the smallest squared frequency;

(2) $\le$ the largest squared frequency;

(3) a good approximation to ω_k^2 if $\underline{y}$ is an approximation to $\underline{u}^{(k)}$.

(Formally, $\dfrac{\bar{V}}{\bar{T}}$ is *stationary* near each mode.)

Governing equations for continuous systems

Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $w(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\frac{\partial^2 w}{\partial t^2} - P\frac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \frac{1}{2}P\int\left(\frac{\partial w}{\partial x}\right)^2 dx$	$T = \frac{1}{2}m\int\left(\frac{\partial w}{\partial t}\right)^2 dx$

Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $f(x,t)$ per unit length.
Polar moment of area is $J = (\pi/2)\left(a^4 - b^4\right)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\frac{\partial^2 \theta}{\partial t^2} - GJ\frac{\partial^2 \theta}{\partial x^2} = f(x,t)$	$V = \frac{1}{2}GJ\int\left(\frac{\partial \theta}{\partial x}\right)^2 dx$	$T = \frac{1}{2}\rho J\int\left(\frac{\partial \theta}{\partial t}\right)^2 dx$

Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $w(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\frac{\partial^2 w}{\partial t^2} - EA\frac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \frac{1}{2}EA\int\left(\frac{\partial w}{\partial x}\right)^2 dx$	$T = \frac{1}{2}\rho A\int\left(\frac{\partial w}{\partial t}\right)^2 dx$

Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $w(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\frac{\partial^2 w}{\partial t^2} + EI\frac{\partial^4 w}{\partial x^4} = f(x,t)$	$V = \frac{1}{2}EI\int\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx$	$T = \frac{1}{2}\rho A\int\left(\frac{\partial w}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.

ENGINEERING TRIPOS PART IIA

Module 3C6 Examination, 2009

Answers

1 (a) $\cos\alpha L\cosh\alpha L = -1$, (b) $\omega^2 = \omega_1^2 + \dfrac{E}{12\rho}\left[\dfrac{\int_0^L 3h_0^2\,\delta h\left(u_1''\right)^2 dx - \omega_1^2\int_0^L \delta h\, u_1^2\, dx}{\int_0^L h_0\, u_1^2\, dx}\right]$.

2 (b) $\omega_n = \dfrac{1}{L}\sqrt{\dfrac{E}{\rho}}\left(n-\dfrac{1}{2}\right)\pi, \quad u_n(x) = K\sin\dfrac{(n-1/2)\pi x}{L}; \quad n = 1, 2, 3\ldots$

(c) $\left.\dfrac{dg}{dt}\right|_{x=L/4} = \dfrac{2}{\rho A L}\sum_{n=1,2..}(-1)^n \sin\dfrac{(n-1/2)\pi}{4}\cos\omega_n t$; (d) $\dfrac{3L}{4}\sqrt{\dfrac{\rho}{E}}$.

3 (a) $V = \dfrac{P}{2L}\left[y_1^2 + (y_2-y_1)^2 + (y_3-y_2)^2 + (y_4-y_3)^2 + y_4^2\right]$

$T = \dfrac{m}{2}\left[\dot{y}_1^2 + 2\dot{y}_2^2 + 2\dot{y}_3^2 + \dot{y}_4^2\right]$

(d) $\alpha = 1.78, -0.28$; $\omega_1^2 = 0.219\dfrac{P}{Lm}$.

4 (a) $\omega_1 = 0, \quad u^{(1)} = [1 \quad 1 \quad 1]^T; \quad \omega_2 = \sqrt{\dfrac{k}{m}}, \quad u^{(1)} = [1 \quad 0 \quad -1]^T;$

$\omega_3 = \sqrt{\dfrac{2k}{m}}, \quad u^{(1)} = [1 \quad -1 \quad 1]^T$

(b) $-\Delta, -180°$ (c) $\dot{y}(0^+) = \dfrac{I}{4m}$ (d) $y_2\left(\sqrt{\dfrac{m}{k}}\right) = 0.075\dfrac{I}{\sqrt{mk}}$.