ENGINEERING TRIPOS PART IIA

Thursday 30 April 2009 9.00 to 10.30

Module 3C7

MECHANICS OF SOLIDS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
3C7 Mechanics of Solids datasheet (2 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 A flywheel is to be used as an energy storage device. It consists of a thin circular annulus, of outer radius R and inner radius αR ($0.1 \le \alpha \le 0.9$), and has a uniform thickness t. The disk material is of density ρ, Young's modulus E and Poisson ratio ν. During operation, the disk spins at a constant angular velocity ω, and has negligible support on its inner and outer boundaries.

(a) Determine the mass m and kinetic energy K of the flywheel. [20%]

(b) Obtain the distributions of radial stress σ_{rr} and hoop stress $\sigma_{\theta\theta}$ that satisfy the boundary conditions, making use of the datasheet as appropriate. [15%]

(c) Obtain an expression for the spin speed ω_Y to give first yield according to the Tresca criterion. [20%]

(d) The flywheel is spun at the speed ω_Y. Assuming that the Poisson ratio equals zero, what value of α maximises the energy density K/m? [30%]

(e) Suggest other designs that might increase the value of K/m. [15%]

2 (a) The dam shown in Fig. 1 is to be modelled as an isotropic elastic wedge with an apex angle of 90°. Show that the Airy stress function

$$\phi = r^{\lambda+1}\left[A\cos(\lambda+1)\theta + B\cos(\lambda-1)\theta\right]$$

can satisfy the boundary conditions on the upstream and downstream faces. Hence calculate the values for A, B and λ. [45%]

(b) Show that the stresses in the dam due to the pressure of the water only are given by [25%]

$$\sigma_{rr} = \frac{-\rho g r}{4\sqrt{2}}(\cos\theta - \cos 3\theta)$$

$$\sigma_{\theta\theta} = \frac{-\rho g r}{4\sqrt{2}}(3\cos\theta + \cos 3\theta)$$

$$\sigma_{r\theta} = \frac{-\rho g r}{4\sqrt{2}}(\sin\theta + \sin 3\theta)$$

(c) Sketch a Mohr's circle for this stress state and show that when θ = 30° one of the principal stresses is zero. [30%]

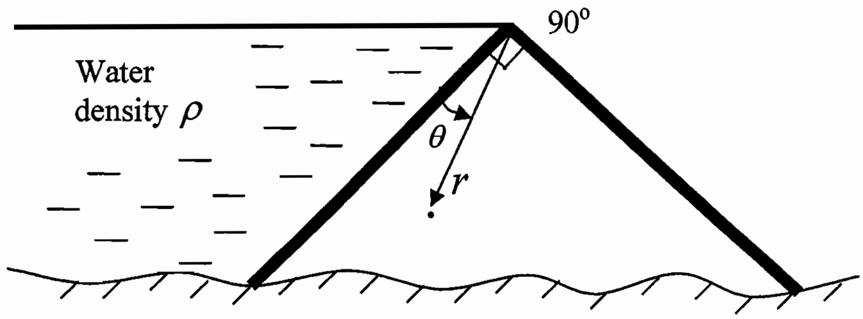

Fig. 1

3 An elastic fan blade has an uniform crescent-shaped cross-section composed of two circular arcs as shown in Fig. 2. The blade is twisted by a torque T about a longitudinal axis that is normal to the section.

(a) Show that a suitable Prandtl stress function for shear stress in the blade is

$$\phi = C\left(1 - \frac{b^2}{r^2}\right)\left(r^2 - 2ar\cos\theta\right)$$

where C is a constant and $a < b < 2a$. [40%]

(b) If the blade is twisted through an angle of α per unit length about the longitudinal axis, find the location and magnitude of the maximum shear stress on both the concave and convex surfaces of the section. [40%]

(c) Explain how you would obtain the St Venant's torsional stiffness of the blade, without doing any explicit calculations. [20%]

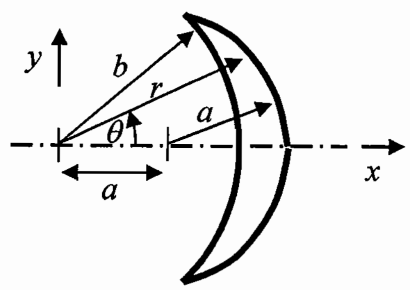

Fig. 2

4 (a) Explain briefly why the Upper Bound theorem is useful in modelling metal forming and shaping operations. [15%]

(b) Figure 3 shows a machining process in which a work-piece is driven at speed v past a sharp cutting tool so that a continuous chip is removed. The depth of cut is t, and the tool-face has an inclination α, as shown. Deformation occurs at a single shear discontinuity, inclined at angle ϕ. The chip moves up the face of the tool and initially this interface can be considered frictionless. The material can be taken to be rigid-perfectly-plastic with shear flow stress k. Use an upper bound analysis to confirm that the most likely value of ϕ is given by

$$\phi = \frac{\pi}{4} + \frac{\alpha}{2} \ .$$

[45%]

(c) Friction between the cutting tool and the chip can be included by supposing that over an interfacial length ℓ, the chip is resisted by a shear stress of magnitude k. For the case that $\alpha = 0$ determine a plausible estimate for ϕ in terms of ℓ and t. [40%]

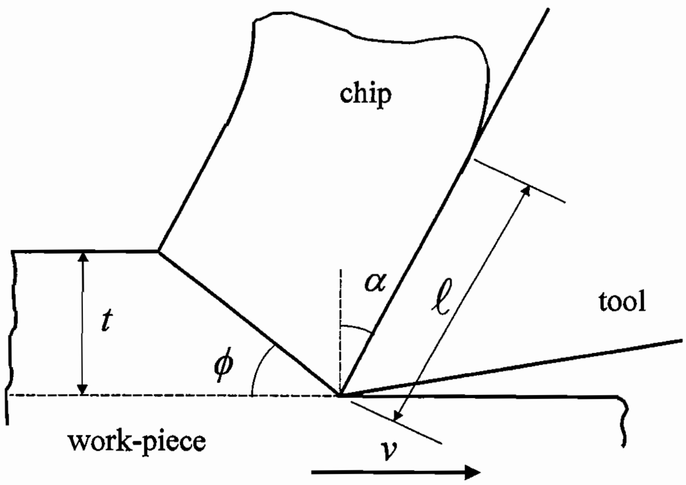

Fig. 3

END OF PAPER

Module 3C7: Mechanics of Solids

ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

	Discs and tubes	Spheres
Equilibrium	$\sigma_{\theta\theta} = \dfrac{d(r\sigma_{rr})}{dr} + \rho\omega^2 r^2$	$\sigma_{\theta\theta} = \dfrac{1}{2r}\dfrac{d(r^2\sigma_{rr})}{dr}$
Lamé's equations (in elasticity)	$\sigma_{rr} = A - \dfrac{B}{r^2} - \dfrac{3+\nu}{8}\rho\omega^2 r^2 - \dfrac{E\alpha}{r^2}\int_c^r rT\,dr$	$\sigma_{rr} = A - \dfrac{B}{r^3}$
	$\sigma_{\theta\theta} = A + \dfrac{B}{r^2} - \dfrac{1+3\nu}{8}\rho\omega^2 r^2 + \dfrac{E\alpha}{r^2}\int_c^r rT\,dr - E\alpha T$	$\sigma_{\theta\theta} = A + \dfrac{B}{2r^3}$

2. Plane stress and plane strain

	Cartesian coordinates	Polar coordinates
Strains	$\varepsilon_{xx} = \dfrac{\partial u}{\partial x}$	$\varepsilon_{rr} = \dfrac{\partial u}{\partial r}$
	$\varepsilon_{yy} = \dfrac{\partial v}{\partial y}$	$\varepsilon_{\theta\theta} = \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$
	$\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$	$\gamma_{r\theta} = \dfrac{\partial v}{\partial r} + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r}$
Compatibility	$\dfrac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \dfrac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \dfrac{\partial^2 \varepsilon_{yy}}{\partial x^2}$	$\dfrac{\partial}{\partial r}\left\{r\dfrac{\partial \gamma_{r\theta}}{\partial \theta}\right\} = \dfrac{\partial}{\partial r}\left\{r^2\dfrac{\partial \varepsilon_{\theta\theta}}{\partial r}\right\} - r\dfrac{\partial \varepsilon_{rr}}{\partial r} + \dfrac{\partial^2 \varepsilon_{rr}}{\partial \theta^2}$
or (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}(\sigma_{xx} + \sigma_{yy}) = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}\right\}(\sigma_{rr} + \sigma_{\theta\theta}) = 0$
Equilibrium	$\dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} = 0$	$\dfrac{\partial}{\partial r}(r\sigma_{rr}) + \dfrac{\partial \sigma_{r\theta}}{\partial \theta} - \sigma_{\theta\theta} = 0$
	$\dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \sigma_{xy}}{\partial x} = 0$	$\dfrac{\partial \sigma_{\theta\theta}}{\partial \theta} + \dfrac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$
$\nabla^4\phi = 0$ (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}\left\{\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2}\right\} = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}\right\} \cdot \left\{\dfrac{\partial^2 \phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2}\right\} = 0$
Airy Stress Function	$\sigma_{xx} = \dfrac{\partial^2 \phi}{\partial y^2}$	$\sigma_{rr} = \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2}$
	$\sigma_{yy} = \dfrac{\partial^2 \phi}{\partial x^2}$	$\sigma_{\theta\theta} = \dfrac{\partial^2 \phi}{\partial r^2}$
	$\sigma_{xy} = -\dfrac{\partial^2 \phi}{\partial x \partial y}$	$\sigma_{r\theta} = -\dfrac{\partial}{\partial r}\left\{\dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta}\right\}$

3. Torsion of prismatic bars

Prandtl stress function: $\sigma_{zx}\ (= \tau_x) = \dfrac{d\psi}{dy}, \quad \sigma_{zy}\ (= \tau_y) = -\dfrac{d\psi}{dx}$

Equilibrium: $T = 2\int_A \psi\, dA$

Governing equation for elastic torsion: $\nabla^2\psi = -2G\beta$ where β is the angle of twist per unit length.

4. Total potential energy of a body

$$\Pi = U - W$$

where $U = \frac{1}{2}\int_V \underset{\sim}{\varepsilon}^T [D]\, \underset{\sim}{\varepsilon}\, dV$, $W = \underset{\sim}{P}^{\,T} \underset{\sim}{u}$ and $[D]$ is the elastic stiffness matrix.

5. Principal stresses and stress invariants

Values of the principal stresses, σ_P, can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx} - \sigma_P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma_P & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma_P \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_P.

Expanding: $\sigma_P^3 - I_1\sigma_P^2 + I_2\sigma_P - I_3 = 0$ where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}.$$

6. Equivalent stress and strain

Equivalent stress $\bar{\sigma} = \sqrt{\frac{1}{2}}\ \{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\}^{1/2}$

Equivalent strain increment $d\bar{\varepsilon} = \sqrt{\frac{2}{3}}\ \{d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2\}^{1/2}$

7. Yield criteria and flow rules

Tresca

Material yields when maximum value of $|\sigma_1 - \sigma_2|$, $|\sigma_2 - \sigma_3|$ or $|\sigma_3 - \sigma_1| = Y = 2k$, and then,

if σ_3 is the intermediate stress, $d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

von Mises

Material yields when, $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{d\varepsilon_1}{\sigma'_1} = \frac{d\varepsilon_2}{\sigma'_2} = \frac{d\varepsilon_3}{\sigma'_3} = \frac{d\varepsilon_1 - d\varepsilon_2}{\sigma_1 - \sigma_2} = \frac{d\varepsilon_2 - d\varepsilon_3}{\sigma_2 - \sigma_3} = \frac{d\varepsilon_3 - d\varepsilon_1}{\sigma_3 - \sigma_1} = \lambda = \frac{3}{2}\frac{d\bar{\varepsilon}}{\bar{\sigma}}.$$