ENGINEERING TRIPOS PART IIA

Thursday 7 May 2009 2.30 to 4

Module 3D7

FINITE ELEMENT METHODS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: Data sheet (3 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) For the quadratic bar element shown in Fig. 1:

(i) give the shape functions and the $\boldsymbol{B}$ matrix for this element; [30%]

(ii) if the stiffness EA varies quadratically along the element according to

$$EA = E_0 A_0 (1 + x^2)$$

compute the first row of the element stiffness matrix using Gauss numerical integration. Use sufficient integration points so that the integration is exact. [40%]

(b) For a problem which involves 10000 of the quadratic bar elements, if the entries in the stiffness matrix are stored using double precision (8 bytes per entry), estimate the computer memory required to store:

(i) all terms (zero and non-zero) in the stiffness matrix; [10%]

(ii) only the non-zero terms in the stiffness matrix. [20%]

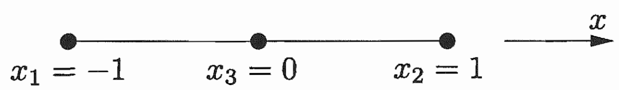


Fig. 1

2 (a) Figure 2(a) shows one six-noded triangular element for which the normal heat fluxes are prescribed between the nodes 1 and 3 to be $-\overline{q}$ and between the nodes 1 and 2 to be $\overline{q}$.

(i) Sketch the shape functions for nodes 1 and 4 and obtain analytical expressions for them in the (x, y) coordinate system. [20%]

(ii) Compute all six components of the element source vector. [40%]

(b) Figure 2(b) shows a five-node triangular element to be used as a transition element.

(i) Sketch the shape function of node 2. No computation is required. [15%]

(ii) Give an analytical expression for the shape function of node 1. [25%]

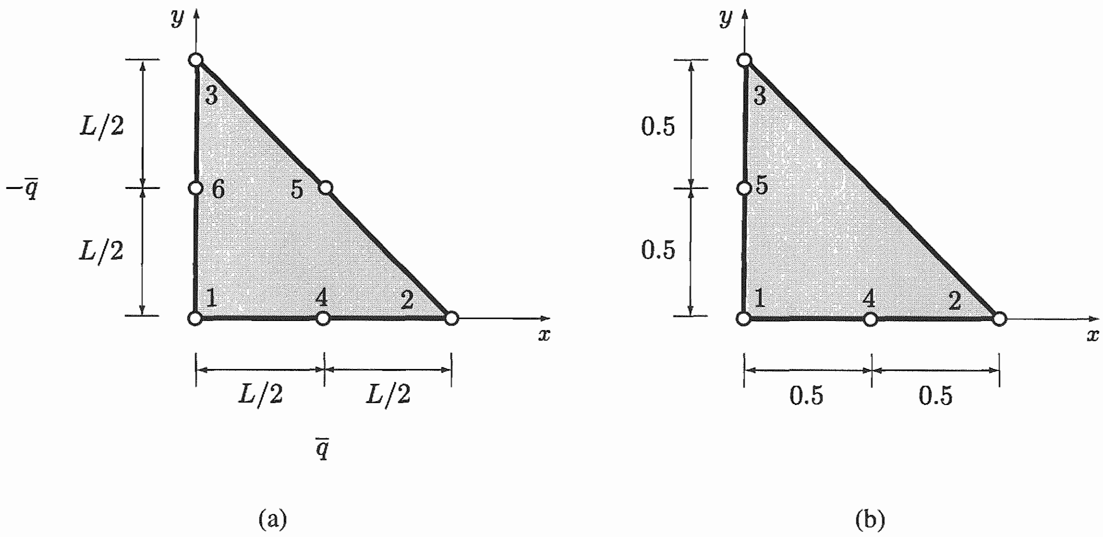


Fig. 2

3 Figure 3 shows a four-noded iso-parametric element and the corresponding parent element. A point P in the isoparametric element corresponds to the quadrature point $(1/\sqrt{3}, 1/\sqrt{3})$.

(a) Compute the coordinates of the point P in the isoparametric element. [30%]

(b) Evaluate the Jacobian matrix $\boldsymbol{J}$ at point P:

$$\boldsymbol{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

[35%]

(c) The displacements at node 3 are $u_x = 0.01$ and $u_y = 0.02$ and all the other nodal displacements are equal to zero. Compute the strain components ε_{xx}, ε_{xy} and ε_{yy} at point P.

[35%]

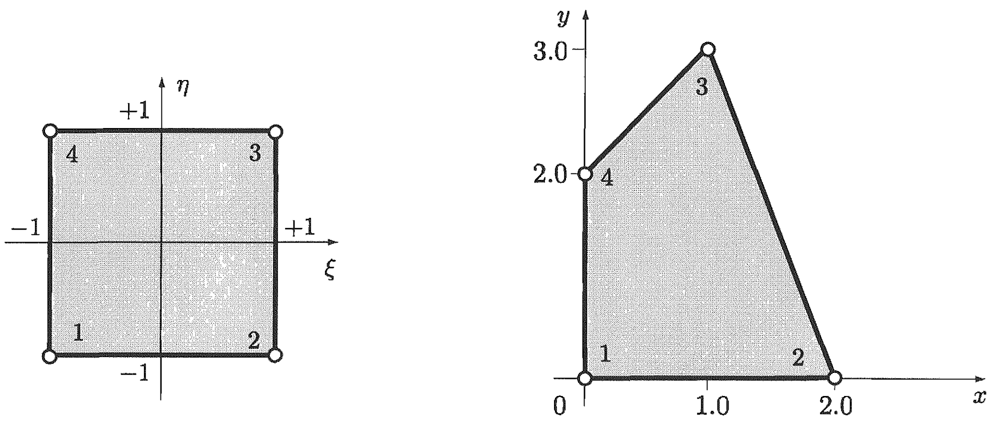


Fig. 3

4 (a) Consider a beam modelled by two two-node beam elements as shown in Fig. 4 which use Hermitian shape functions. The consistent element mass matrix for an element which runs from x_1 to x_2 is given by

$$\boldsymbol{M}_e = \int_{x_1}^{x_2} \rho \boldsymbol{N}^T \boldsymbol{N}\, dx$$

For constant density ρ:

(i) What is the dimension of the global mass matrix for this problem? [15%]

(ii) Compute a global mass matrix using the mass lumping technique. Do you expect the mass lumping approach to be suitable for a dynamic analysis of this problem? Motivate your answer. [30%]

(iii) Why is a lumped mass matrix sometimes desirable? [15%]

(b) We wish to solve a time-dependent heat conduction problem. The matrix form of the finite element formulation is

$$\boldsymbol{M}\dot{\boldsymbol{a}} + \boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}$$

(i) Using the Leapfrog scheme,

$$\boldsymbol{a}_{n+1} - \boldsymbol{a}_{n-1} = 2\Delta t \dot{\boldsymbol{a}}_n$$

formulate the problem such that it can be solved for $\boldsymbol{a}_{n+1}$. [20%]

(ii) Is the Leapfrog scheme implicit or explicit, and is it self-starting? [20%]

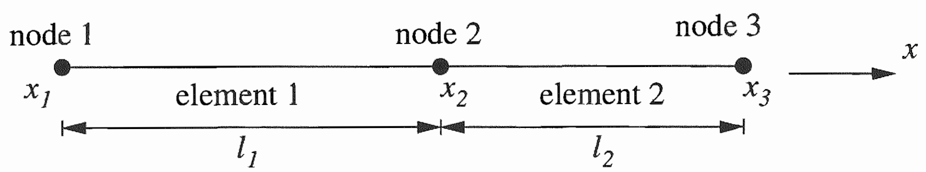


Fig. 4

END OF PAPER

3D7 DATA SHEET

Element relationships Elasticity

Displacement	$u = Na_e$
Strain	$\varepsilon = Ba_e$
Stress (2D/3D)	$\sigma = D\varepsilon$
Element stiffness matrix	$k_e = \int_{V_e} B^T DB\, dV$
Element force vector (body force only)	$f_e = \int_{V_e} N^T f\, dV$

Heat conduction

Temperature	$T = Na_e$
Temperature gradient	$\nabla T = Ba_e$
Element conductance matrix	$k_e = \int_{V_e} B^T DB\, dV$

Beam bending

Displacement	$v = Na_e$
Curvature	$\kappa = Ba_e$
Element stiffness matrix	$k_e = \int_{V_e} B^T EIB\, dV$

Elasticity matrices

2D plane strain

$$D = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

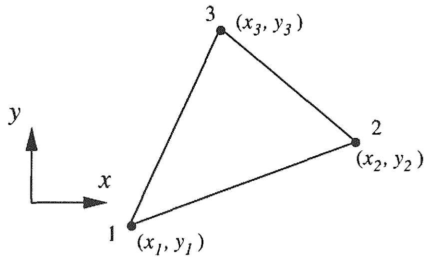
2D plane stress

$$D = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

Heat conductivity matrix (2D)

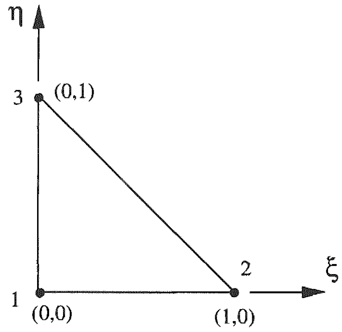
$$D = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

Shape functions

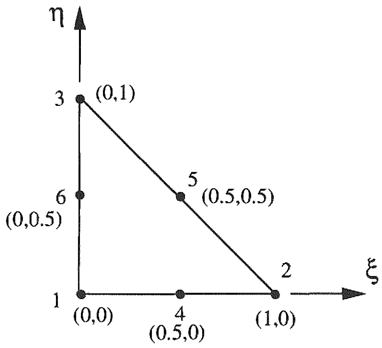


$$N_1 = ((x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y)/2A$$
$$N_2 = ((x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y)/2A$$
$$N_3 = ((x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y)/2A$$

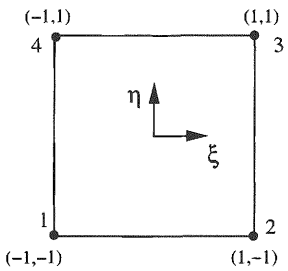
A = area of triangle



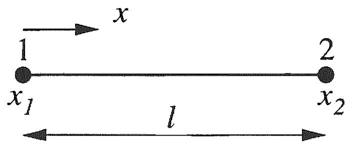
$$N_1 = 1 - \xi - \eta$$
$$N_2 = \xi$$
$$N_3 = \eta$$



$$N_1 = 2(1 - \xi - \eta)^2 - (1 - \xi - \eta)$$
$$N_2 = 2\xi^2 - \xi$$
$$N_3 = 2\eta^2 - \eta$$
$$N_4 = 4\xi(1 - \xi - \eta)$$
$$N_5 = 4\eta\xi$$
$$N_6 = 4\eta(1 - \xi - \eta)$$



$$N_1 = (1 - \xi)(1 - \eta)/4$$
$$N_2 = (1 + \xi)(1 - \eta)/4$$
$$N_3 = (1 + \xi)(1 + \eta)/4$$
$$N_4 = (1 - \xi)(1 + \eta)/4$$



Hermitian element

$$N_1 = \frac{-(x-x_2)^2(-l+2(x_1-x))}{l^3}$$

$$M_1 = \frac{(x-x_1)(x-x_2)^2}{l^2}$$

$$N_2 = \frac{(x-x_1)^2(l+2(x_2-x))}{l^3}$$

$$M_2 = \frac{(x-x_1)^2(x-x_2)}{l^2}$$

Gauss integration in one dimension on the domain $(-1,1)$ Using n Gauss integration points, a polynomial of degree $2n-1$ is integrated exactly.

number of points n	location ξ_i	weight w_i
1	0	2
2	$-\frac{1}{\sqrt{3}}$	1
	$\frac{1}{\sqrt{3}}$	1
3	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
	0	$\frac{8}{9}$
	$\sqrt{\frac{3}{5}}$	$\frac{5}{9}$

The list of numerical answers for 3D7

1. (a)

$$k_{11} = 1.733 E_0 A_0$$
$$k_{12} = 0.4 E_0 A_0$$
$$k_{13} = -2.133 E_0 A_0$$

   (b) (i) $3.2 \cdot 10^9$ bytes; (ii) $6.4 \cdot 10^5$ bytes

2. (a) (ii)

$$\boldsymbol{f}^e = \begin{bmatrix} 0 & -\frac{1}{6}\overline{q}L & \frac{1}{6}\overline{q}L & -\frac{2}{3}\overline{q}L & 0 & \frac{2}{3}\overline{q}L \end{bmatrix}^T$$

3. (a) $x = 0.9554$ $y = 2.1994$

   (b)

$$\boldsymbol{J} = \begin{bmatrix} 0.6057 & 0.3943 \\ -0.3943 & 1.3943 \end{bmatrix}$$

   (c) $\epsilon_{xx} = 3.943 \cdot 10^{-3}$, $\epsilon_{yy} = 7.886 \cdot 10^{-3}$, $\epsilon_{xy} = 5.915 \cdot 10^{-3}$