ENGINEERING TRIPOS PART IIA

Tuesday 28 April 2009 9 to 10.30

Module 3F1

SIGNALS AND SYSTEMS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) A periodic sequence $\{v_k\}$ $(k \geq 0)$ is given by

$$v_k = \begin{cases} f_k & 0 \leq k \leq n-1 \\ v_{k-n} & k \geq n \end{cases}$$

If $F(z)$ denotes the z-transform of the finite sequence $f_0, f_1, ..., f_{n-1}$ show that the z-transform of $\{v_k\}$ is given by

$$V(z) = \frac{F(z)}{1 - z^{-n}}$$

[20%]

(b) The input sequence $\{u_k\}$ and the output sequence $\{y_k\}$ for a digital filter satisfy the difference equation

$$y_k = \beta y_{k-n} + u_k - u_{k-n}, \quad k \geq 0$$

where $|\beta| < 1$.

(i) Calculate the transfer function of this filter. [20%]

(ii) Find the pole and zero positions of the filter. Sketch their locations assuming $\beta > 0$ for the cases: $n = 1$ and $n = 2$. [20%]

(iii) Calculate the response of the filter to the periodic signal $\{v_k\}$ in part (a), and sketch the form of the response. [20%]

(iv) Find the step response of the filter.
[Hint: chose a suitable f_k and use part (b)(iii)] [20%]

2 (a) The block diagram of a discrete-time feedback control system is shown in Fig. 1, where α and β are constants satisfying $\alpha > 0$ and $\beta > 0$.

(i) Determine the range of values for k such that the closed-loop system is stable as a function of α and β. [15%]

(ii) For the range of values of k found in (i), find $\lim_{k \to \infty} e_k$ when the input $\{u_k\}$ is a unit step. [25%]

(iii) With $\alpha = 1$, what values of β and k result in a steady-state error of zero? [10%]

(b) A linear system with impulse response $h(t)$ is excited by a white-noise random process, $X(t)$, with autocorrelation function $r_{XX}(t) = \delta(t)$.

(i) Show that the auto-correlation function of the output of the system $Y(t)$ is given by

$$r_{YY}(t) = h(t) * h(-t)$$

Hence, write down an expression for the power spectral density of $Y(t)$, $S_Y(\omega)$, in terms of $H(\omega)$, the frequency response of the system. [35%]

(ii) Describe how the magnitude of the frequency response of a plant may be estimated in practice while keeping it functioning online and excited by random perturbations. [15%]

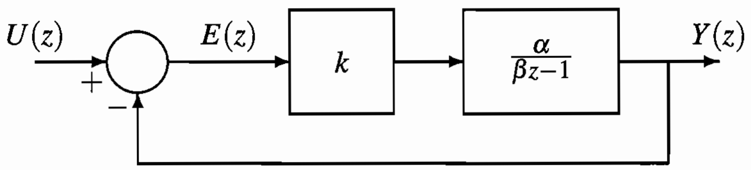

Fig. 1

(TURN OVER

3 (a) If independent random processes X and Y have probability density functions (pdfs) $f_X(x)$ and $f_Y(y)$, and there is a process Z, which is the sum of X and Y, then show that the pdf, $f_Z(z)$, is given by the convolution of f_X with f_Y. [25 %]

(b) The characteristic function of X is defined to be $\Phi_X(u) = E[e^{jux}]$, where $E[\cdot]$ is the expectation operator. Show that the characteristic function of Z is given by

$$\Phi_Z(u) = \Phi_X(u) \cdot \Phi_Y(u)$$

[25 %]

(c) If the pdfs of X and Y are given by

$$f_X(x) = \frac{1}{2x_0} e^{-|x|/x_0} \quad \text{and} \quad f_Y(y) = \frac{1}{2y_0} e^{-|y|/y_0}$$

where $x_0 > 0$ and $y_0 > 0$, derive an expression for $\Phi_Z(u)$. [25 %]

(d) Briefly summarise (without producing the full proof) how characteristic functions may be used to prove the central limit theorem for the case where all the random inputs have the same pdfs. [25 %]

4 (a) The mutual information $I(X;Y)$ between random processes X and Y is defined as the difference between the entropy of X and the conditional entropy of X, given Y. If X and Y are both discrete processes with N_X and N_Y states respectively, derive an expression for $I(X;Y)$ in terms of the probability mass functions (pmfs), $p_X(x_i)$ and $p_Y(y_j)$, and the conditional pmf $p_{X|Y}(x_i|y_j)$ for $i = 1,...,N_X$ and $j = 1,...,N_Y$. Hence, using Bayes rule, show that $I(Y;X)$ is the same as $I(X;Y)$. [35 %]

(b) A symmetric binary communications channel has a probability of bit error of 0.15 . Calculate the mutual information between the binary input of the channel X and the binary output Y, if the probability of X being $+1$ is β and of it being -1 is $(1-\beta)$. [30 %]

(c) Determine the value of β which maximises the mutual information and obtain the theoretical capacity of the channel. To convey 1000 bits of user information with negligible probability of error over this channel, estimate the number of bits that would need to be transmitted, assuming that a very good error-correcting code can be used. [35 %]

END OF PAPER