ENGINEERING TRIPOS PART IIA

Thursday 7 May 2009 9 to 10.30

Module 3F3

SIGNAL AND PATTERN PROCESSING

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 The Discrete Time Fourier Transform (DTFT) of a sequence $\{x_n\}$, $n = 0, \pm 1, \pm 2, \ldots, \pm\infty$ is given by

$$X(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} x_n e^{-jn\omega T}$$

where T is the sample period.

(a) Explain why the DTFT cannot in practice be evaluated on a digital computer and how this problem can be overcome. Show how the Discrete Fourier Transform (DFT)

$$X_p = \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi np}{N}}$$

where N is the transform size, can be derived directly from the DTFT expression by application of a rectangular window function sequence $\{w_n\}$, $n = 0, \pm 1, \pm 2, \ldots, \pm\infty$

$$w_n = \begin{cases} 1, & \text{for } n = 0, 1, \ldots, N-1 \\ 0, & \text{otherwise} \end{cases}$$

and discretization of the frequency variable. [30%]

(b) Show that the relationship between the DFT spectrum values X_p and the true DTFT spectrum $X(e^{j\omega T})$ may be expressed as a frequency domain convolution of the true spectrum and the spectrum $W(e^{j\theta})$ of the window function

$$\frac{1}{2\pi} \int_0^{2\pi} W(e^{j\theta})\, X\Big(e^{j(\omega T - \theta)}\Big)\, d\theta$$

Determine an expression for the window spectrum $W(e^{j\theta})$. [30%]

(c) Explain how the radix-2 Fast Fourier Transform (FFT) algorithm allows very efficient implementation of the DFT above when N is a power of 2. Your description should include: the number of stages in the radix-2 FFT algorithm and the number of "butterfly" computations required in relation to the transform size N; the "butterfly" structure and the computations required to compute one "butterfly" operation; bit reversal operations and in-place computation. Show that the total number of real operations (that is, real multiplications, additions or subtractions) required is approximately $5N\log_2 N$. [40%]

2 (a) List the advantages and disadvantages of IIR filters in comparison with FIR filters. [20%]

(b) Describe how the bilinear transform

$$s = \frac{1 - z^{-1}}{1 + z^{-1}}$$

can be used to convert analogue filters to digital filters. State the type of filters that can usefully be designed using this method, and any distortions that are introduced. [20%]

(c) A bandpass filter is required in a channelised radio communications system with sampling rate 8kHz and 3dB cutoff frequencies of 2kHz and 3kHz. We are interested in designing such a filter from an analogue lowpass prototype given by

$$H(s) = \frac{1}{1 + s}$$

which has a 3dB cutoff frequency of 1 rad/sec.

Using the lowpass to bandpass transformation

$$s' = \frac{s^2 + \omega_l \omega_u}{s(\omega_u - \omega_l)}$$

with lower cutoff at ω_l and upper cutoff at ω_u together with the bilinear transform, design the required digital filter. [40%]

(d) The filter is to be implemented in fixed precision digital hardware. Explain briefly some potential undesirable consequences of this implementation, and suggest strategies for overcoming them. [20%]

(TURN OVER

3 The autoregressive moving-average (ARMA) model is a wide sense stationary process $\{x_n\}$ satisfying the equation:

$$\sum_{p=0}^{P} a_p x_{n-p} = \sum_{q=0}^{Q} b_q w_{n-q}$$

where $\{w_n\}$ is a white process with unity variance, and without loss of generality we assume $a_0 = b_0 = 1$.

(a) By interpreting $\{x_n\}$ as a result of filtering and assuming that the poles of the ARMA process all lie within the unit circle, derive the power spectrum of the above ARMA process. [30%]

(b) Consider the following stationary random process as a particular case of the ARMA model presented above:

$$x_n = \sum_{q=0}^{Q} b_q w_{n-q}$$

where $\{w_n\}$ is a white process with unity variance, and without loss of generality we assume $b_0 = 1$. What type of process is this? Show that the autocorrelation function of this process is given by

$$r_{XX}[k] = \sum_{q=0}^{Q} b_q r_{XW}[k-q]$$

where $r_{XW}[k]$ is the crosscorrelation between x_n and w_n at lag k. [20%]

(c) Further show that the autocorrelation above can be expressed as: [20%]

$$r_{XX}[k] = \begin{cases} \sum_{l=0}^{Q-k} b_{l+k} b_l, & \text{for } k = 0, 1, \ldots, Q \\ 0, & \text{for } k > Q \end{cases}$$

(d) Given the following estimates of the autocorrelation function of a particular signal:

$$\begin{aligned} r_{XX}[1] &= -\frac{1}{2} \\ r_{XX}[k] &= 0 \text{ for } |k| > 1 \end{aligned}$$

determine the coefficient values for this particular model and roughly sketch the signal power spectrum. [30%]

4 Consider the k-means clustering algorithm which seeks to minimise the cost function

$$C = \sum_{n=1}^{N} \sum_{k=1}^{K} s_{nk} \|x_n - m_k\|^2$$

where m_k is the mean (centre) of cluster k, x_n is data point n, $s_{nk} = 1$ signifies that data point n is assigned to cluster k, and there are N data points and K clusters.

(a) Given all the assignments $\{s_{nk}\}$, derive the value of m_k which minimises the cost C and give an interpretation in terms of the k-means algorithm. [30%]

(b) Give a probabilistic interpretation of k-means and describe how it can be generalised to unequal cluster sizes (number of data points per cluster) and non-spherical (elongated) clusters as shown in Fig. 1 below. [30%]

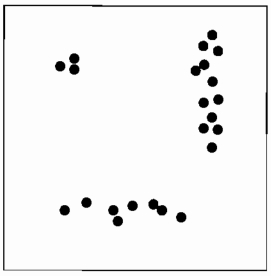

Fig. 1

(c) In many real-world applications, data points arrive sequentially and one wants to cluster them as they come in. Devise a sequential variant of the k-means algorithm which takes in one data point at a time and updates the means $\{m_1, \ldots, m_K\}$ sequentially without revisiting previous data points. Describe your sequential algorithm. [40%]

END OF PAPER