1

Crib1 (a) $w(z) = Uz - \frac{i\Gamma}{2\pi}\ln(z) + \frac{m}{2\pi}\ln(z)$

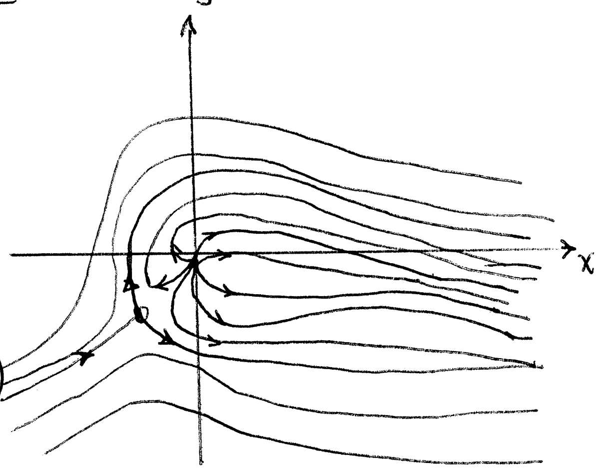
$$= \frac{z}{2\pi} + \frac{i}{2\sqrt{2}\pi}\ln(z) + \frac{1}{2\sqrt{2}\pi}\ln(z)$$

(b) $\frac{dw}{dz} = \frac{1}{2\pi}\left(1 + \frac{i}{\sqrt{2}z} + \frac{1}{\sqrt{2}z}\right) = 0$

$$\frac{1}{\sqrt{2}z}(i+1) = -1$$

Stagnation point $z = \frac{1}{\sqrt{2}}(1+i)$

(c) streamline pattern



(d) Find the streamline that represents the surface

$$w(z) = \phi + i\psi = Ux + Uyi - \frac{i\Gamma}{2\pi}(\ln(r) + i\theta) + \frac{m}{2\pi}(\ln(r) + i\theta)$$

$$\psi = Uy - \frac{\Gamma}{2\pi}\ln(r) + \frac{m\theta}{2\pi} = \frac{y}{2\pi} + \frac{\ln(r)}{2\sqrt{2}\pi} + \frac{\theta}{2\sqrt{2}\pi}$$

Now go to the stagnation point $\begin{cases} r = 1 \\ \theta = 5\pi/4 \\ y = -\frac{1}{\sqrt{2}} \end{cases}$

$$\psi = -\frac{1}{2\pi\sqrt{2}} + \frac{5\pi}{8\sqrt{2}\pi} = \frac{1}{2\sqrt{2}\pi}\left(\frac{5\pi}{4} - 1\right)$$

Cribl

Now look for downstream where $\theta \to 0$ on top

$\theta \to 2\pi$ on bottom.

Top: $\frac{1}{2\sqrt{2}\pi}\left(\frac{5\pi-4}{4}\right) = \frac{y_T}{2\pi} + \frac{\ln(\sqrt{x^2+y_T^2})}{2\sqrt{2}\pi} + \frac{0}{2\sqrt{2}\pi}$ ①

Bottom: $\frac{1}{2\sqrt{2}\pi}\left(\frac{5\pi-4}{4}\right) = \frac{y_B}{2\pi} + \frac{\ln\sqrt{x^2+y_B^2}}{2\sqrt{2}\pi} + \frac{2\pi}{2\sqrt{2}\pi}$ ②

At large x $\sqrt{x^2+y_T^2} \approx x$, $\sqrt{x^2+y_B^2} \approx x$

② − ① $\frac{1}{2\pi}(y_B - y_T) + \frac{1}{\sqrt{2}} = 0$

$y_T - y_B = \sqrt{2}\pi$

Crib 2

(a) $w(z) = -\frac{1}{2\pi}\ln(z) + \frac{1}{4\pi}\ln(z-1)$

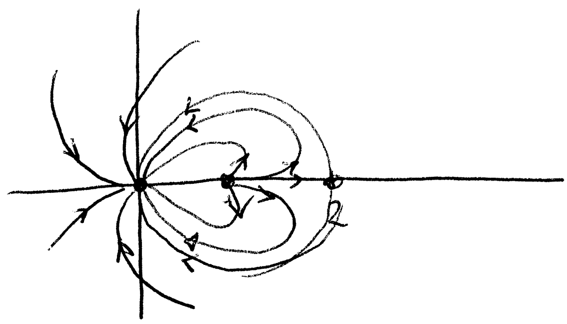
(b) $\frac{dw}{dz} = -\frac{1}{2\pi z} + \frac{1}{4\pi(z-1)} = 0$

$$-4\pi(z-1) + 2\pi z = 0$$

Stagnation point: $z = 2$

~~(c)~~

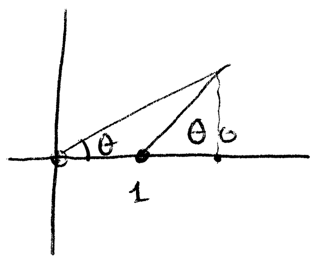
Streamline pattern



(c) Downstream beyond the stagnation point is safe $x > 2$.

To work out above and below we need the stagnation s/L.

$$\psi = -\frac{\theta}{2\pi} + \frac{\theta_0}{4\pi}$$



At stagnation point, $\theta = 0$, $\theta_0 = 0$, so $\psi = 0$.

Now above and below the source, $\theta = \pm\frac{\pi}{2}$.

$-\frac{\theta}{2\pi} \pm \frac{1}{8} = 0$, $\theta = \cancel{\theta}\pm\frac{\pi}{4} \Rightarrow y = \pm 1$.

Crib2

(d) Now add a uniform flow.

$$w(z) = Uz - \frac{1}{2\pi}\ln(z) + \frac{1}{4\pi}\ln(z-1)$$

stagnation point.

$$\frac{dw}{dz} = U - \frac{1}{2\pi z} + \frac{1}{4\pi(z-1)} = 0$$

$$4\pi U z^2 - (4\pi U + 1) z + 2 = 0$$

$$z = \frac{4\pi U + 1 \pm \sqrt{(4\pi U+1)^2 - 32\pi U}}{8\pi U}$$

For z to be real:

$$(4\pi U + 1)^2 - 32\pi U \geq 0$$

or $$16\pi^2 U^2 - 24\pi U + 1 \geq 0$$

$$\Rightarrow \quad U < 0.0139$$

$$U > 0.4635$$

For $U \approx 0.0139$, $z \approx 3.36$ (beyond source.

For $U = 0.4635$, $z = 0.5858$ (between sink & source)

So $U = 0.4635$ is no good. and the maximum strength is $U = 0.0139$.

Qib 3

(a) Potential

$$w(z) = Uz + \frac{8\pi U a^2}{2\pi(z-ia)} + \frac{8\pi U a^2}{2\pi(z+ia)}$$

$$= Uz + \frac{4Ua^2}{z-ia} + \frac{4Ua^2}{z+ia}$$

(b) stagnation point $\frac{dw}{dz} = 0$

$$\frac{dw}{dz} = U - \frac{4Ua^2}{(z-ia)^2} - \frac{4Ua^2}{(z+ia)^2}$$

$$= U - \frac{8Ua^2(z^2-a^2)}{(z^2+a^2)^2} = 0$$

$$\Rightarrow \quad z^4 - 6a^2z^2 + 9a^4 = 0$$

$$(z^2-3a^2)^2 = 0 \Rightarrow \cancel{z} = \pm\sqrt{3}\,a$$

(c) length along x-axis is $2\sqrt{3}a$.

(d) $w(z) = \phi + i\psi = Uz + \dfrac{8Ua^2 z}{z^2+a^2}$

at stagnation point $z = \sqrt{3}a$, stream function $\psi(\sqrt{3}a) = 0$.

Suppose at ib, $\psi(ib) = 0$, $b \neq 0$,

$$\cancel{\phi}\, w(ib) = iUb + \frac{8Ua^2(ib)}{a^2-b^2} = ibU\left(1 + \frac{8a^2}{a^2-b^2}\right)$$

$$\psi(ib) = bU\left(1 + \frac{8a^2}{a^2-b^2}\right) = 0 \Rightarrow b = \pm 3a.$$

$\Rightarrow$ body width normal to flow at $x = 0$

is $6a$.

Cribs to 3A1, 2010

Jie Li

Crib 4

(a) In Cartesian coordinate systems,

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.$$

Hence

$$\omega = -\left(\frac{\partial^2 \psi}{\partial^2 x} + \frac{\partial^2 \psi}{\partial y^2}\right) = -\nabla^2 \psi.$$

(b) Because there is no θ variation, $\nabla^2 \psi = \frac{1}{r}\frac{d}{dr}(r\frac{d\psi}{dr})$, we have,

$$\begin{cases} \psi = -\frac{1}{4}\Omega r^2 + A \ln r + B, & r < a, \\ \psi = C \ln r + D, & r > a, \end{cases}$$

where A, B, C, and D are 4 constants.

(c) Choose to have $\psi = 0$ on $r = a$, and reject the singularity at $r = 0$ by choosing $A = 0$, we get

$$\begin{cases} \psi = -\frac{1}{4}\Omega(r^2 - a^2), & r < a, \\ \psi = C \ln(r/a) +, & r > a, \end{cases}$$

If we choose to match u_θ at $r = a$, we end up with

$$\begin{cases} \psi = -\frac{1}{4}\Omega(r^2 - a^2), & r < a, \\ \psi = -\frac{1}{2}\Omega a^2 \ln(r/a), & r > a, \end{cases}$$

(d) Let $a \to 0$ and $\Omega \to \infty$ in Rankine's vortex in such a ~~wya~~ way that $-\frac{1}{2}\Omega a^2 = \frac{\Gamma}{\pi}$. The inner region shrinks to nothing, and in the outer region

$$\psi = \frac{\Gamma}{2\pi} \ln(r/a).$$

So a line vortex is a very thin Rankine vortex.

Q1. Solution Crib5

a) The pressure gradient must be zero since very near the wall the pressure-gradient is balanced by the viscous terms (the inertia terms go to zero)

$$\underbrace{U\frac{dU}{dx} + V\frac{dU}{dy}}_{\substack{\text{Inertia terms}\\ \to 0 \text{ near the}\\ \text{wall since } U\to 0\\ \text{and } V\to 0}} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{d^2U}{dy^2}$$

$$\frac{1}{\rho}\frac{dp}{dx} = \nu\frac{d^2U}{dy^2} \quad \text{as } y \to 0$$

$$U = U_1\left(\frac{4}{3}\frac{y}{\delta} - \frac{1}{3}\left(\frac{y}{\delta}\right)^4\right)$$

$$\frac{dU}{dy} = U_1\left(\frac{4}{3}\frac{1}{\delta} - \frac{4}{3}\frac{y^3}{\delta^4}\right)$$

$$\frac{d^2U}{dy^2} = U_1\left(-4\frac{y^2}{\delta^4}\right) = \frac{-4U_1}{\delta^2}\left(\frac{y}{\delta}\right)^2 \quad \text{as } y \to 0$$

$$\frac{d^2U}{dy^2} \to 0$$

hence $\frac{1}{\rho}\frac{dp}{dx} \to 0$

Q1 solution cont'd. Crib 5

b) (i) $\delta^* = \int_0^\delta 1 - \frac{U}{U_1}\,dy$

$$= \delta \int_0^1 1 - \frac{U_t}{U_1}\,d\eta$$

$$= \delta \int_0^1 1 - \tfrac{4}{3}\eta + \tfrac{1}{3}\eta^4\,d\eta$$

$$= \delta \left[\eta - \tfrac{2}{3}\eta^2 + \tfrac{1}{15}\eta^5\right]_0^1$$

$$= \delta \left[\frac{15 - 10 + 1}{15}\right) = \frac{6}{15}\delta = \frac{2}{5}\delta$$

(ii) $\theta = \delta \int_0^1 \left(1 - \frac{U}{U_1}\right)\frac{U}{U_1}\,d\eta$

$$= \delta \int_0^1 \left(1 - \tfrac{4}{3}\eta + \tfrac{1}{3}\eta^4\right)\left(\tfrac{4}{3}\eta - \tfrac{1}{3}\eta^4\right)d\eta$$

$$= \delta \int_0^1 \tfrac{4}{3}\eta - \tfrac{1}{3}\eta^4 - \tfrac{16}{9}\eta^2 + \tfrac{4}{9}\eta^5 + \tfrac{4}{9}\eta^5 - \tfrac{1}{9}\eta^8\,d\eta$$

$$= \delta \left[\tfrac{2}{3}\eta^2 - \tfrac{16}{27}\eta^3 - \tfrac{1}{15}\eta^5 + \tfrac{4}{27}\eta^6 - \tfrac{1}{81}\eta^9\right]_0^1$$

$$= \delta \left[\cancel{54 \cdots 8}\ \frac{270 - 240 - 27 + 60 - 5}{405}\right) = \frac{58}{405}\delta$$

iii) $H = \delta^*/\theta = \frac{81}{29}$

iv) $\tau_w = \mu \frac{\partial U}{\partial y}\Big|_{y=0} = \mu \frac{4}{3}\frac{U_1}{\delta}$ $\qquad C_f' = \frac{\tau_w}{\frac{1}{2}\rho U_1^2} = \frac{4\mu U_1}{3\delta \frac{1}{2}\rho U_1^2}$

$$= \frac{8\nu}{3U_1\delta} = \frac{8}{3}\frac{\nu}{U_1\delta}$$

Q1 solution cont'd, Crib 5

c) ZPG $\frac{d\theta}{dx} = \frac{C_f'}{2}$

$$\theta = \frac{58}{405}\delta \qquad \frac{d\theta}{dx} = \frac{58}{405}\frac{d\delta}{dx} = \frac{4}{3}\frac{\nu}{U_1\delta}$$

Note that U_1 = constant since it is ZPG

$$\int_0^\delta \frac{58}{405}\delta\, d\delta = \int_0^x \frac{4}{3}\frac{\nu}{U_1}\,dx$$

$$\frac{58}{405}\times\frac{1}{2}\delta^2 = \frac{4}{3}\frac{\nu x}{U_1}$$

$$\frac{29}{405}\delta^2 = \frac{4}{3}\frac{\nu x}{U_1}$$

$$\delta = \sqrt{\frac{405}{29}\frac{4}{3}\frac{\nu x}{U_1}}$$

$$C_f' = \frac{8}{3}\frac{\nu}{U_1\delta}$$

$$\frac{U_1\delta}{\nu} = \frac{U_1}{\nu}\sqrt{\frac{405\cdot 4}{29\cdot 3}\frac{\nu x}{U_1}}$$

$$= \sqrt{\frac{405}{29}\frac{4}{3}\frac{U_1 x}{\nu}}$$

$$C_f' = \frac{8}{3}\frac{1}{\sqrt{\frac{405}{29}\frac{4}{3}\frac{U_1 x}{\nu}}} = \frac{8}{3}\frac{1}{\sqrt{\frac{405}{29}\frac{4}{3}}}\frac{1}{\sqrt{\frac{U_1 x}{\nu}}}$$

Q4 cont'd **Crib6** (1)

in this limit find an expression for the skin friction coefficient C_f'

Solution

a) $U = \frac{U_\tau}{K} \ln\left(\frac{y}{h}\right) + C\left(\frac{hU_\tau}{\nu}\right)$

$\frac{dU}{dy} = \frac{U_\tau}{Ky} + 0$ as required

b) $\frac{U_1}{U_\tau} = \frac{1}{K} \ln\left(\frac{\delta}{h}\right) + C\left(\frac{hU_\tau}{\nu}\right)$

$U_\tau = \sqrt{\frac{\tau_w}{\rho}}$ $\tau_w = \rho U_\tau^2$ $\frac{\tau_w}{\frac{1}{2}\rho U_1^2} = \frac{2U_\tau^2}{U_1^2} = C_f'$

$\frac{C_f'}{2} = \frac{U_\tau^2}{U_1^2}$ $\frac{U_1}{U_\tau} = \sqrt{\frac{2}{C_f'}}$

$$\sqrt{\frac{2}{C_f'}} = \frac{1}{K} \ln\left(\frac{\delta}{h}\right) + C\left(\frac{hU_\tau}{\nu}\right)$$

c) $\frac{U_1}{U_\tau} = \frac{1}{K} \ln\left(\frac{\delta}{h}\right) + C\left(\frac{hU_\tau}{\nu}\right)$

$\frac{U}{U_\tau} = \frac{1}{K} \ln\left(\frac{y}{h}\right) + C\left(\frac{hU_\tau}{\nu}\right)$

$$\frac{U_1}{U_\tau} - \frac{U}{U_\tau} = \frac{1}{K} \ln\left(\frac{\delta}{y}\right) = -\frac{1}{K} \ln\left(\frac{y}{\delta}\right)$$

indep of h and h^+ as required

Q4 Soln Cont'd. Crib 6

d) $\delta = ah$

$$\sqrt{\frac{2}{C_f'}} = \frac{1}{\kappa}\ln\left(\frac{\delta}{a\delta}\right) + C\left(\frac{a\delta U_\tau}{\nu}\right)$$

$$= \frac{1}{\kappa}\ln\left(\frac{1}{a}\right) + C\left(a\frac{\delta U_\tau}{\nu}\right)$$

But $C \rightarrow C_0$

$$\sqrt{\frac{2}{C_f'}} = \frac{-1}{\kappa}\ln(a) + C_0$$

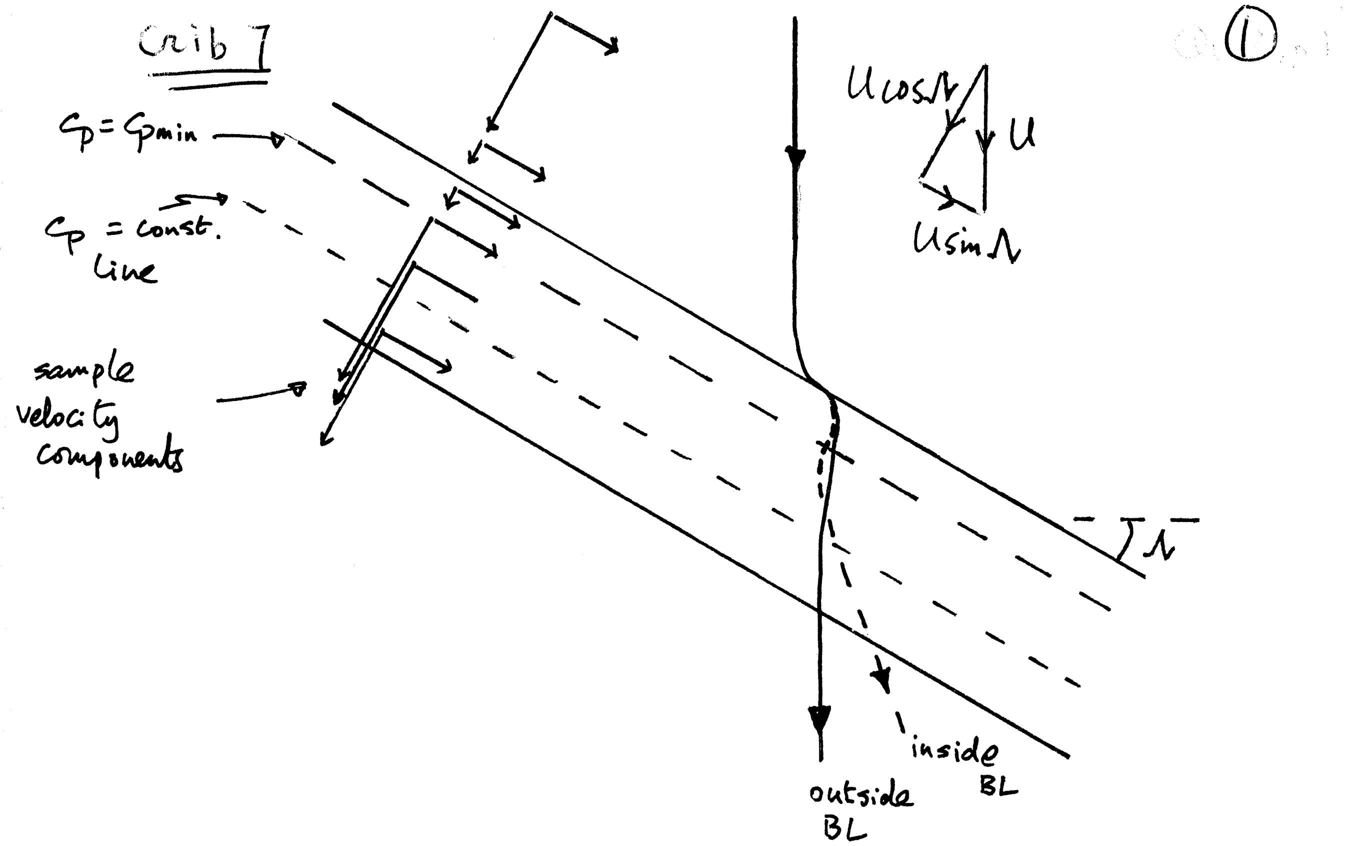
$$\frac{2}{C_f'} = \left(-\frac{1}{\kappa}\ln(a) + C_0\right)^2$$

$$C_f' = \frac{2}{\left(C_0 - \frac{1}{\kappa}\ln(a)\right)}$$

which is constant!

Crib 7

a)



- shape of streamline outside BL follows from velocities, on the basis that wing-normal component varies as for 2D aerofoil, while wing-parallel component remains constant;
- flow in BL is subject to same pressures as that outside, but at lower velocity, which decreases further as BL is traversed ⇒ hence cross-streamline pressure gradients cause greater (and increasing) curvature of external streamline.

b) (i) Away from root and tip, expect streamlines to look similar to infinite wing case. At root, streamlines must be in main flow direction (symmetry). At tip, decreased loading will result in smaller curvatures / deflections.

(ii) Calculate downwash velocities at collocation points:

$$\frac{v_{dA}}{U} = \frac{v_{dAS}}{\Gamma_A}\frac{\Gamma_A}{US} + \frac{v_{dAS}}{\Gamma_B}\frac{\Gamma_B}{US} + \frac{v_{dAS}}{\Gamma_C}\frac{\Gamma_C}{US} = \underline{\underline{0.1379}}$$

Similarly, $v_{dB}/U = 0.1380$, $v_{dC}/U = 0.1380$

Hence the values of $\Gamma_A, \Gamma_B, \Gamma_C$ are consistent, and $\alpha = \underline{\underline{0.138\text{ rad}}}$

b) (iii) Lift on an element of the unswept wing is $\rho U \Gamma \Delta s$.

——————— " ——————— swept wing is $\rho (U\cos\Lambda)\Gamma \frac{\Delta s}{\cos\Lambda} = \rho U \Gamma \Delta s$.

Hence we can compare Γ_s directly to comment on lift distributions. Effect of sweepback is to increase wing tip loading and decrease root loading.

c) Stall will initiate further towards tip of swept wing, for two reasons:

- outboards shift of wing loading, along with unchanged chord distribution, implies that maximum local C_l also found further outboard;
- swept-wing BL is more prone to separation, but root symmetry constraint mitigates this problem in the inboard region.

Measures to alter behaviour (given fixed sweep angle):

- at design stage, reduce outboard loading by increasing chord or twisting wing down there; alternatively it might be possible to use a different aerofoil section with a higher stall C_L;
- post hoc, add wing fence(s) or similar device(s) (eg vortilon, saw-tooth leading edge).

Crib 8

a) $y_c = 2Y\frac{x}{c}\left(1-\frac{x}{c}\right) \Rightarrow \frac{dy_c}{dx} = \frac{2Y}{c}\left(1-\frac{2x}{c}\right)$

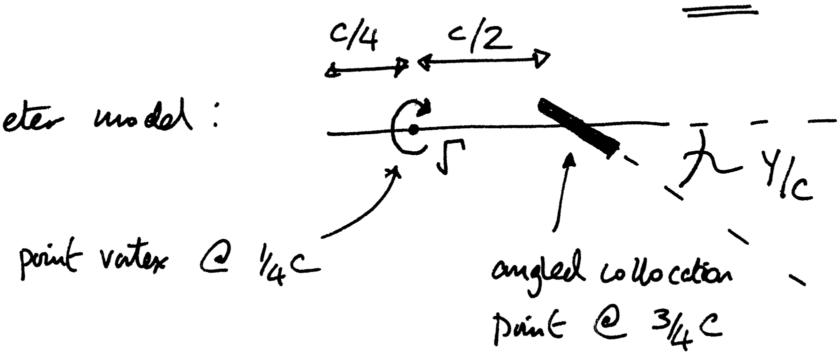
$$= -\frac{2Y}{c}\cos\theta, \text{ where } x = c\,\frac{1+\cos\theta}{2}$$

So $-2\frac{dy_c}{dx} = 4\frac{Y}{c}\cos\theta$; Fourier coefficients (by inspection) are $g_0 = 0,\ g_1 = 4\frac{Y}{c},\ g_2 = g_3 = \ldots = 0$

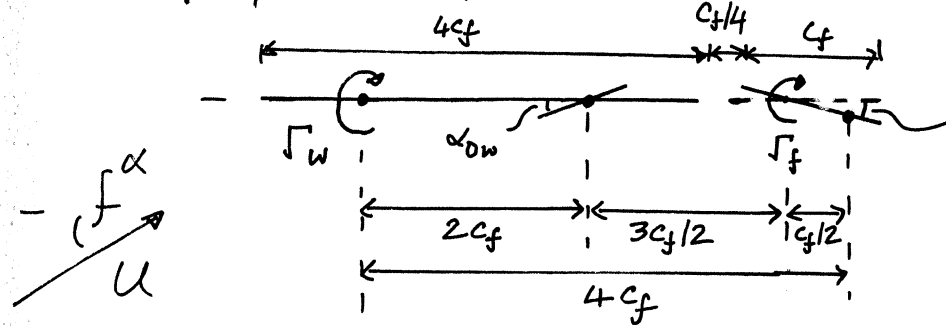
Lift @ zero incidence : $C_{L0} = \pi\left(g_0 + g_1/2\right) = 2\pi\frac{Y}{c}$

For zero lift : $2\pi\alpha + C_{L0} = 0 \Rightarrow \underline{\underline{\alpha = -Y/c}}$

Lumped-parameter model :



b) Lumped-parameter representation :



Note that small δ_f approx. required for distances here.

Conditions for no net upwash at collocation points :

$$\left.\begin{aligned} U(\alpha - \alpha_{0w}) &= \frac{\Gamma_w}{2\pi . 2c_f} - \frac{\Gamma_f}{2\pi . 3c_f/2} \\ U(\alpha + \delta_f) &= \frac{\Gamma_w}{2\pi . 4c_f} + \frac{\Gamma_f}{2\pi . c_f/2} \end{aligned}\right\} \text{ solve for } \Gamma_w, \Gamma_f$$

Crib8

Result : $\Gamma_w = \pi U c_f \left[\frac{32}{7}\alpha - \frac{24}{7}\alpha_{0w} + \frac{8}{7}\delta_f\right]$

$$\Gamma_f = \pi U c_f \left[\frac{3}{7}\alpha + \frac{3}{7}\alpha_{0w} + \frac{6}{7}\delta_f\right]$$

$$c_{lw} = \frac{\rho U \Gamma_w}{\frac{1}{2}\rho U^2 . 4c_f} = 2\pi\left[\frac{8}{7}\alpha - \frac{6}{7}\alpha_{0w} + \frac{2}{7}\delta_f\right]$$

$$c_{lf} = \frac{\rho U \Gamma_f}{\frac{1}{2}\rho U^2 c_f} = 2\pi\left[\frac{3}{7}\alpha + \frac{3}{7}\alpha_{0w} + \frac{6}{7}\delta_f\right]$$

c) Dependence on α: c_{lw} increases, as would expect. $\partial c_{lw}/\partial\alpha$ is $> 2\pi$ (the single aerofoil value) because α increase also leads to more lift on flap, and resulting upwash on wing increases its loading. Conversely, $\partial c_{lf}/\partial\alpha$ is markedly less than 2π, because of wing downwash.

Dependence on α_{0w}: c_{lw} increases with decreasing α_{0w}, but by factor $6/7$ of single-wing result. This is because rise in wing lift increases downwash on flap $\rightarrow$ c_{lf} drops ($\partial c_{lf}/\partial\alpha_{0w}$ +ve) $\rightarrow$ flap upwash on main wing decreases, offsetting the 'expected' lift gain.

Dependence on δ_f : c_{lf} increases, but by factor $6/7$ of single-wing result, because resulting upwash on wing increases c_{lw} (note $\partial c_{lw}/\partial\delta_f$ +ve), leading to some additional downwash on flap.

Numerics Answers

Jie Li

Quesion 1,
(b) Stagnation point $z = (1+i)/\sqrt{2}$.
(d) Body width $\sqrt{2}\pi$.
Question 2,
(b) Stagnation point $z = 2$
(c) $y = \pm 1$. $x > 2$.
(d) Maximum velocity $U \approx 0.0139$.
Question 5,
(b) (iii) $H = 81/29$.
Question 7,
(b) (ii) $\alpha \approx 0.138$.