2010 PART IIA 3A6 HEAT AND MASS TRANSER
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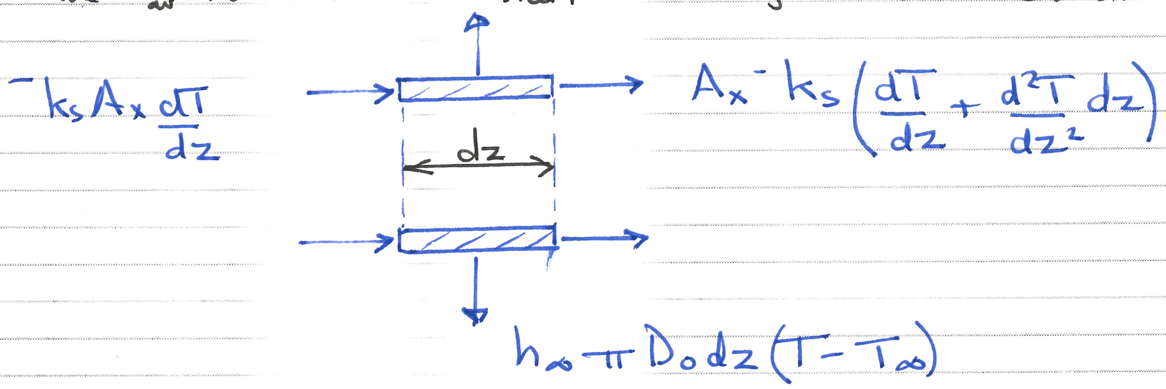
ENGINEERING TRIPOS

Paper 3A6

Q 1(a)

Because T_{air} inside is same as T_{steel}, we can ignore internal heat transfer



For an elemental control volume of length dz

$$-k_s \frac{dT}{dz} A_x + k_s \left(\frac{dT}{dz} + \frac{d^2T}{dz^2} dz\right) A_x - h_\infty \pi D_o dz (T - T_\infty) = 0$$

$$\Rightarrow \quad k_s A_x \frac{d^2T}{dz^2} = h_\infty \pi D_o (T - T_\infty)$$

where $A_x = \frac{\pi}{4}(D_o^2 - D_i^2)$

Let $\theta = \dfrac{T - T_\infty}{T_o - T_\infty}$ & $m^2 = \dfrac{4 h_\infty D_o}{k_s (D_o^2 - D_i^2)}$

$\therefore$

$$\frac{d^2\theta}{dz^2} = m^2 \theta$$

General solution is of the form $\theta = Ae^{mz} + Be^{-mz}$

In this case : $z = 0 \quad \theta = 1 \quad \Rightarrow \quad A + B = 1$
$z = \infty \quad \theta = 0 \quad \Rightarrow \quad Ae^{m\infty} + Be^{-m\infty} = 0$

$\therefore A = 0 \Rightarrow B = 1$

$\therefore$ Solution is

$$\theta = e^{-mz}$$

as required

Q1(b)

$L = 0.1$ $\quad T = 400\,k$ $\quad k_s = 10\,W/mK$

$D_o = 4\times10^{-3}\,m$ $\quad T_o = 900\,k$

$D_i = 3\times10^{-3}\,m$ $\quad T_\infty = 300\,k$

$$\therefore \quad m^2 = \frac{4h_\infty}{k_s D_o (1 - D_i^2/D_o^2)} = \frac{4h_\infty}{10 \cdot 4\times10^{-3}(1-0.75^2)}$$

$$= 228.57\,h_\infty$$

$$\theta = \frac{T - T_\infty}{T_o - T_\infty} = \frac{400-300}{900-300} = \frac{1}{6}$$

$$\therefore \quad \frac{1}{6} = e^{-\sqrt{228.57 h_\infty}\,\cdot\,0.1}$$

$$\ln\frac{1}{6} = \sqrt{228.57}\sqrt{h_\infty}\cdot 0.1$$

$$\Rightarrow \quad \underline{\underline{h_\infty = 1.4046}}$$

(c) The equation for stagnant air in the tube is Laplace's equation for the temperature field $\therefore$

$$\nabla^2 T = 0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}$$

The term $\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0$ due to symmetry.

The boundary condition are

$z = 0 \quad T = T_o$

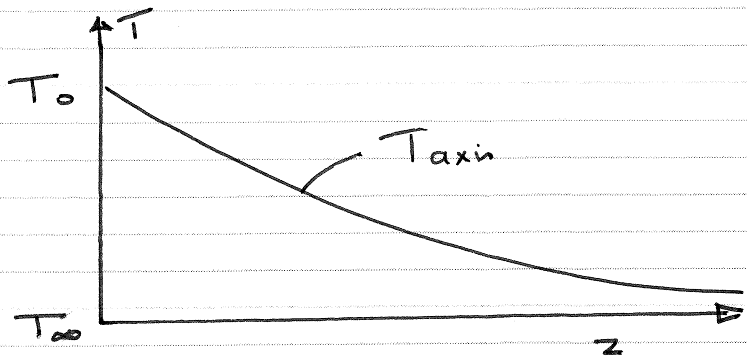
$z = \infty \quad T = T_\infty$

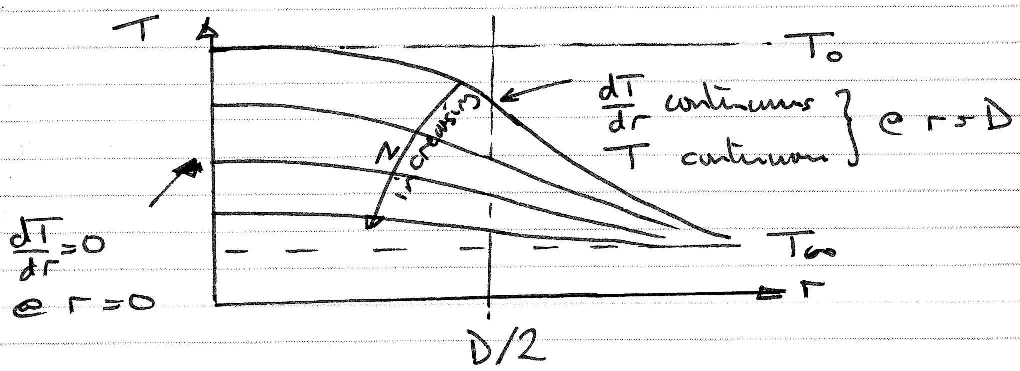
$r = 0 \quad \frac{\partial T}{\partial r} = 0$

$$r = \frac{D}{2} \quad k_a \frac{\partial T}{\partial r}\bigg|_{inside} \left(= k_s \frac{\partial T}{\partial r}\bigg|_{steel}\right) = k_a\frac{\partial T}{\partial r}\bigg|_{outside}$$

$$= h_\infty (T - T_\infty)$$

Q1(d)


T
T_0
T_{axis}
T_∞
z


T
T_0
$\frac{dT}{dr}$ continuous
T continuous
@ r = D
z increasing
$\frac{dT}{dr} = 0$
@ r = 0
T_∞
r
D/2

Q1(e) Approximation of uniform T of the air in the tube is valid if

$$T_0 - T_s \ll T_s - T_\infty$$

where T_s is temperature of the steel tube & T_0 is centreline Temp

Now radial heat flux is given by

$$q_{inside} \simeq k_a \frac{T_0 - T_s}{D/2} \qquad (D = \text{mean dia})$$

$$q_{outside} = h_\infty (T_s - T_\infty)$$

$$\therefore \quad \frac{T_0 - T_s}{T_s - T_\infty} \simeq \frac{h_\infty D}{2k_a} \ll 1$$

Across the tube, uniform T requires

$$T_{si} - T_{so} \ll T_{so} - T_\infty$$

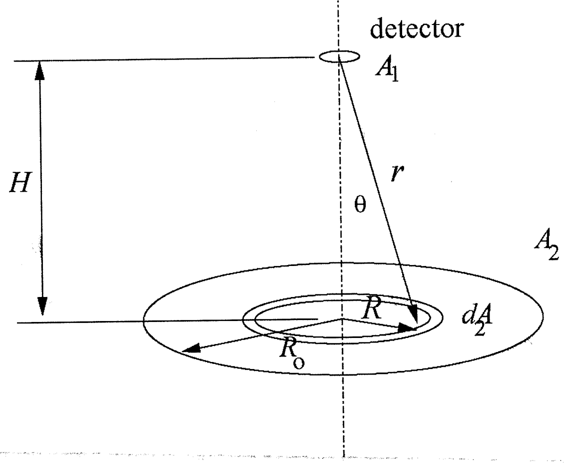
Again,

$$q \simeq k_s \frac{T_{si} - T_{so}}{\frac{1}{2}(D_o - D_i)} \simeq h_\infty (T_s - T_\infty)$$

$$\therefore \quad \frac{T_{si} - T_{so}}{T_{so} - T_\infty} \simeq \frac{h_\infty (D_o - D_i)}{k_s} \simeq \frac{h_\infty t}{k_s} \ll 1$$

where t is the thickness of the tube. This is equivalent to saying that the external resistance is very much greater than the internal resistance to the heat flux.

Q2



Total radiation received by the detector is

$$G_1(\lambda) = \iint \varepsilon I_{b2} \cos\theta_2 \, dA_2 \cos\theta_1 \frac{dA_1}{r^2}$$

$$= \iint \varepsilon_n \cos^2\theta_2 \, I_{b2} \cos\theta_1 \, dA_2 \frac{dA_1}{r^2}$$

Both plates are parallel

$$\therefore \quad \cos\theta_1 = \cos\theta_2$$

& $\qquad A_2 \gg A_1$

$\therefore$

$$G_1(\lambda) = \varepsilon_n I_{b2} A_1 \int_{A_1} \cos^3\theta \frac{dA_2}{r^2} \qquad \text{where } dA_2 = 2\pi R\,dR$$

Now $\quad \cos\theta = \frac{H}{r} \quad$ & $\quad r^2 = H^2 + R^2$

$$\therefore \cos^3\theta = \frac{H^3}{(H^2+R^2)^{3/2}}$$

$\therefore$

$$G(\lambda) = \varepsilon_n I_{b2} A_1 \int_0^{R_o} \frac{H^3}{(H^2+R^2)^{5/2}} 2\pi R\,dR$$

Q2(a) cont

Integrating:

$$G(\lambda) = \varepsilon_n I_{b_2} A_1 \pi H^3 \left[\frac{(H^2 + R^2)^{-3/2}}{-3/2} \right]_0^{R_0}$$

$$= \varepsilon_n I_{b_2} A_1 \frac{2}{3}\pi \cdot \left[\left\{ 1 + \left(\frac{R}{H}\right)^2 \right\}^{-3/2} \right]_0^{R_0}$$

$$= \varepsilon_n I_{b_2} A_1 \frac{2}{3}\pi \left[1 - \frac{1}{(1+(R_0/H)^2)^{3/2}} \right]$$

qed

(b) Ratio is $\frac{G_1(\lambda_0, T)}{G_1(\lambda_0 T_0)} = \frac{\varepsilon_n(\lambda_0, T)\, I_b(\lambda_0, T)}{\varepsilon_n(\lambda_0, T_0\, I_b(\lambda_0, T_0)}$

since all geometric factors remain constant.

Now given:

$$\varepsilon_n = \varepsilon_0 (1 - e^{-\alpha/\lambda T})$$

$$I_b = C_1 / (\pi \lambda^2 (e^{C_2/\lambda T} - 1))$$

$$\therefore \frac{G_1(\lambda_0 T)}{G_{1_0}(\lambda_0, T_0)} = \frac{1 - \exp(-\alpha/(\lambda_0 T))}{1 - \exp(-\alpha/(\lambda_0 T_0))} \cdot \frac{(\exp(\frac{C_2}{\lambda_0 T_0}) - 1)}{(\exp(C_2/\lambda_0 T) - 1)}$$

(c) For use as a detector, we need good T dependence over the range of the detector.

Firstly, we must calibrate at a reference T_0. Then, the ratio in (b) above can be used to give the T.

Consider the values given: $C_2 = 1.4388 \times 10^4$ μmK
$\alpha = 1000$ μmK
$T = 300 \rightarrow 1500$ K
$\lambda = 1 \rightarrow 10$ μm

Q 2(c) cont.

We are looking for a sensible dynamic range over $300 \to 1500K$ and need to choose an appropriate wavelength for the filter.

For $\lambda = 1\mu m$; $\frac{G_1(1500)}{G_1(300)} = 2.3 \times 10^{16}$

$\lambda = 5\mu m$ $\frac{G_1(1500)}{G_1(300)} = 646$

$\lambda = 10\mu m$ $\frac{G_1(1500)}{G_1(300)} = 17$

Therefore, a longer wavelength filter – about $5\mu m$ would be appropriate

(d) Assume grey bodies:

1 = detector 2 = disc 3 = surroundings

Consider

$$G_1 = J_1 A_1 F_{11} + J_2 A_2 F_{21} + J_3 A_3 F_{31}$$

Now:

$$A_3 F_{31} = A_1 F_{13}$$

& $F_{11} = 0$ (circular disc)

& $F_{11} + F_{12} + F_{13} = 0$

$$\Rightarrow F_{13} = 1 - F_{12} = 1 - F_{21}\frac{A_2}{A_1}$$

$$\therefore G_1 = 0 + J_2 A_2 F_{21} + J_3 A_1 F_{13}$$

$$= 0 + J_2 A_2 F_{21} + J_3 A_1 \left(1 - F_{21}\frac{A_2}{A_1}\right)$$

$$= (J_2 - J_3) F_{21} A_2 + J_3 A_1$$

⑧

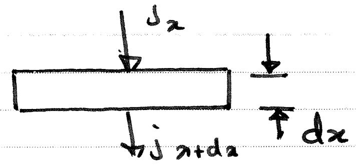
Q2(d) cont.

Therefore, if $T_3 \ll T_2$

then $J_3 \ll J_2 = \varepsilon E_{b2} + (1-\varepsilon) J_3$

where we neglect the contribution from the detector because of its small area. Hence we can neglect the surroundings if at low T.

3(a) Assume fuel concentration is small so that convective flux of oil in reverse direction is small.



Mass conservation for small element of depth dx and area 1:

$$\frac{\partial c}{\partial t}(dx.1) = (j_x - j_{x+dx}).\,1$$

Now $$j_{x+dx} = j_x + \frac{\partial j}{\partial x}dx$$

where $$j = -D\frac{\partial c}{\partial x} \qquad \text{(Fick's Law)}$$

$$\therefore \quad \frac{\partial c}{\partial t}dx = \frac{\partial}{\partial x}\left(D\frac{\partial c}{\partial x}\right)dx$$

$$\Rightarrow \quad \frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} \quad : \quad D = \text{const}$$

(b) Let $$\eta = \frac{x}{2\sqrt{Dt}} \Rightarrow \frac{\partial \eta}{\partial x} = \frac{1}{2\sqrt{Dt}} \;\;\&\;\; \frac{\partial \eta}{\partial t} = \frac{x}{2\sqrt{D}\,t^{3/2}}\cdot\frac{-1}{2}$$
$$= \eta\cdot\frac{-1}{2t}$$

$$\therefore \quad \frac{\partial c}{\partial x} = \frac{\partial c}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{1}{2\sqrt{Dt}}\frac{\partial c}{\partial \eta}$$

$$\frac{\partial^2 c}{\partial x^2} = \frac{\partial}{\partial \eta}\left(\frac{\partial c}{\partial x}\right)\frac{\partial \eta}{\partial x} = \frac{1}{4Dt}\frac{\partial^2 c}{\partial \eta^2}$$

$$\frac{\partial c}{\partial t} = \frac{\partial c}{\partial \eta}\frac{\partial \eta}{\partial t} = \frac{-1}{4}\frac{x}{\sqrt{Dt}.t}\frac{\partial c}{\partial \eta}$$

$$\therefore \quad \frac{-1}{4}\frac{x}{\sqrt{Dt}\,t}\frac{\partial c}{\partial \eta} = \frac{D}{4Dt}\frac{\partial^2 c}{\partial \eta^2}$$

Q3(b) cont

Rearranging

$$\frac{\cancel{4t}}{\cancel{4}}\frac{x}{\sqrt{Dt}}\frac{1}{\cancel{t}}\frac{\partial c}{\partial \eta}+\frac{\partial^2 c}{\partial \eta^2}=0$$

$$2\eta\frac{\partial c}{\partial \eta}+\frac{\partial^2 c}{\partial \eta^2}=0 \qquad \text{qed.}$$

(c) Boundary conditions are

$$c(0) = Sp_{h}$$
$$c(\infty) = 0$$

Let $f(\eta)=\frac{\partial c}{\partial \eta}$

$$\therefore \quad \frac{\partial f}{\partial \eta}+2\eta f=0$$

$$\frac{\partial f}{f}=-2\eta\, d\eta$$

$$\ln f = -\eta^2 + c_1$$

$$f = D_1 e^{-\eta^2} \qquad D = e^{c_1}$$

$$\therefore \quad \frac{\partial c}{\partial \eta}=D_1 e^{-\eta^2}$$

$$c = D_1\int_0^{\eta} e^{-\eta^2}\,d\eta + c(\eta=0)$$

$$= D_1\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(\eta) + c(0).$$

3(c) cont

$$\eta = \infty \Rightarrow 0 = D_1 \frac{\sqrt{\pi}}{2} \cdot 1 + c(0)$$

$$\therefore D_1 = -\frac{2}{\sqrt{\pi}} c(0)$$

Now $c(0) = Sp_h$ = const (told to assume this here)

$$\therefore c(\eta) = Sp_h (1 - \mathrm{erf}(\eta))$$

q.ed.

(d) Fractional rate of change of $p_h = \frac{1}{p_h}\frac{\partial p_h}{\partial t}$

Now $p_h V = m_h RT$ (T; V = const)

$$\therefore \frac{1}{p_h}\frac{dp_h}{dt} = \frac{1}{m_h}\frac{dm_h}{dt} \quad *$$

But $\frac{dm_h}{dt} = +D \frac{\partial c(0,t)}{\partial x}$ (note signs)

Now $\frac{\partial c}{\partial x} = \frac{\partial c}{\partial \eta}\frac{\partial \eta}{\partial x}$; $\frac{\partial \eta}{\partial x} = \frac{1}{2\sqrt{Dt}}$

$$\& \ \frac{\partial c}{\partial \eta} = -Sp_h \exp(-\eta^2)$$

$$\therefore \frac{1}{p_h}\frac{dp_h}{dt} = \frac{-1}{m_h}\frac{1}{2}\sqrt{\frac{D}{t}}\, e^{-\eta^2} S p_h$$

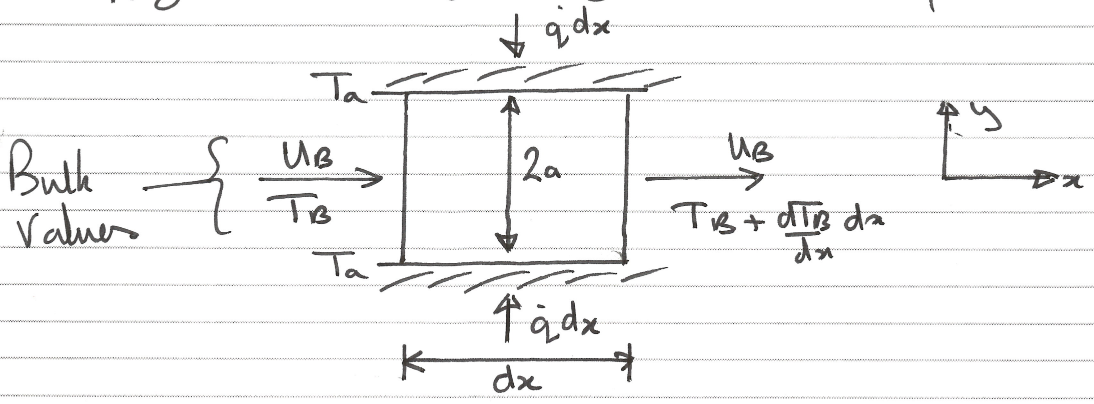
But $\frac{1}{m_h} = \frac{RT}{Vp_h}$

$$\therefore \frac{1}{p_h}\frac{dp_h}{dt} = -\frac{1}{2}\frac{RT}{V}\sqrt{\frac{D}{t}}\, S$$

Approximate is OK if diffusion is relatively slow (see * above) or when head space is large ...

Q4 (a) No axial conduction ∴ only conduction is perpendicular to the flow.

Apply SFEE to following control volume of constant width



$$2\dot{q}\,dx = \dot{m}\,c_p\left(\frac{dT_B}{dx}dx + T_B\right) - \dot{m}\,c_p T_B$$

$$\dot{m} = \rho U_B (2a)$$

$$\therefore \quad \dot{q} = \rho a U_B c_p \frac{dT_B}{dx}$$

(b) Given

$$\frac{\partial T}{\partial x} \neq f(y) \quad \therefore \quad \frac{\partial T}{\partial x} = \frac{\partial T_B}{\partial x} = \frac{dT_B}{dx} = \text{const}$$

$$\therefore \quad \frac{\partial^2 T}{\partial x^2} = 0$$

Flow is fully developed $\therefore \frac{\partial u}{\partial x} = 0$

Continuity eq^n: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$$\therefore \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \quad v = \text{const} = v(y=a) = 0$$

$\therefore \quad \rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$

becomes

$$\rho c_p u \frac{\partial T}{\partial x} = k \frac{\partial^2 T}{\partial y^2}$$

Now $\quad q = \rho a U_B c_p \frac{dT_B}{dx} \quad \& \quad \frac{dT_B}{dx} = \frac{dT}{dx}$

$\therefore \quad \rho c_p u \frac{dT}{dx} = \frac{q}{a} \frac{u}{U_B}$

$\Rightarrow \quad \frac{q}{ak} \frac{u}{U_B} = \frac{\partial^2 T}{\partial y^2}$.

(c) Using the above:

$\frac{\partial^2 T}{\partial y} \sim \frac{\Delta T}{a^2} \quad$ where $\Delta T = T_a - T_0$

$\dot{q} \sim h \Delta T$

$\frac{u}{U_B} \sim 1$

$\therefore \quad \frac{\dot{q}}{ak} \frac{u}{U_B} \sim \frac{h \Delta T}{ak} \cdot 1 \sim \frac{\Delta T}{a^2}$

$\Rightarrow \quad \frac{ha}{k} \sim 1$

(d) $\quad \frac{u}{U_B} = \frac{3}{2} \left(1 - \left(\frac{y}{a} \right)^2 \right)$

Substitute into result from (b) gives

$$\frac{q}{ak} \frac{3}{2} \left(1 - \left(\frac{y}{a} \right)^2 \right) = \frac{\partial^2 T}{\partial y^2}$$

Integrate:

$$\frac{\partial T}{\partial y} = \frac{\dot{q}}{ak} \frac{3}{2} \left(y - \frac{y^3}{3a^2} \right)$$

This satisfies $\frac{\partial T}{\partial y} = 0$ @ $y = 0$ $\therefore$ no constant needed

Q4(d) cont

Integrate again:

$$T = \frac{\dot{q}}{ak}\frac{3}{2}\left(\frac{y^2}{2} - \frac{y^4}{12a^2} + D\right) + C$$

Given that $T = T_a$ when $y = a$

$$C = T_a \quad \& \quad D = -5/_{12}\, a^2$$

$\therefore$

$$T_a - T = \frac{\dot{q}}{ak}\frac{3}{2}\left(\frac{5}{12}a^2 - \frac{y^2}{2} + \frac{y^4}{12a^2}\right)$$

(e) @ $y = 0$; $T = T_o$

$$\therefore \quad T_a - T_o = \frac{\dot{q}}{ak}\frac{3}{2}\left(\frac{5}{12}a^2 + 0 + 0\right)$$

$$= \frac{\dot{q}}{ak}\frac{15}{24}a^2 = \frac{q a}{k}\frac{15}{24}$$

$$Nu = \frac{ha}{k}$$

$$h = \frac{\dot{q}}{T_a - T_o} = \frac{24}{15}\frac{k}{a}$$

$$\therefore Nu = \frac{24}{15}\frac{k}{a}\frac{a}{k} = \frac{24}{15} \sim O(1) \text{ as before}$$

ENGINEERING TRIPOS PART IIA 2010

ANSWERS: MODULE 3A6: HEAT AND MASS TRANSFER

Q1: (b) 1.4 W/m^2K

Q4: (e) 1.6