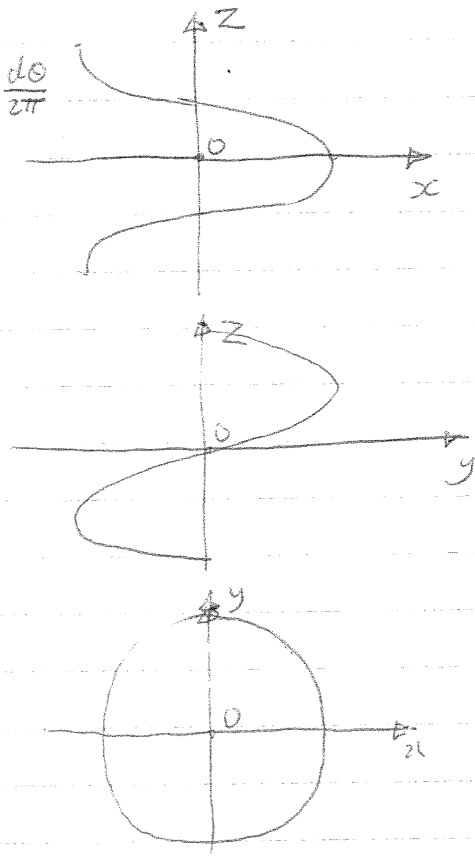
2010 PART IIA 3C5 DYNAMICS
PROFESSOR R S LANGLEY

3C5 2010 CRIBS

1. (a) Three views

$dm = m\dfrac{d\theta}{2\pi}$



Centre of mass is at $(0,0,0)$ by inspection,

$$I_{yy} = \int (x^2+z^2)\,dm = \int_{-\pi}^{\pi}\left(R^2\cos^2\theta + \left(\frac{\theta L}{2\pi}\right)^2\right)\frac{m\,d\theta}{2\pi}$$

$$= \frac{R^2 m}{2\pi}(\pi) + \frac{mL^2}{(2\pi)^3}\left(\frac{2}{3}\pi^3\right)$$

$$= \frac{m}{12}\left(6R^2 + L^2\right)$$

$$I_{xz} = \int xz\,dm = \int_{-\pi}^{\pi} R\cos\theta\,\frac{\theta L}{2\pi}\,m\,\frac{d\theta}{2\pi}$$

$$= \frac{mRL}{4\pi^2}\int_{-\pi}^{\pi}\theta\cos\theta\,d\theta = 0$$

$$I_{xx} = \int (y^2+z^2)\,dm = \frac{m}{12}\left(6R^2+L^2\right) \quad \text{again}$$

$$I_{yz} = \int yz\,dm = \frac{mRL}{4\pi^2}\int_{-\pi}^{\pi}\theta\sin\theta\,d\theta = \frac{2\pi mRL}{4\pi^2} = \frac{mRL}{2\pi}$$

$$I_{zz} = mR^2$$

$$I_{xy} = 0$$

[note: $\int \theta\cos\theta\,d\theta = \theta\sin\theta - \int \sin\theta\,d\theta = \theta\sin\theta + \cos\theta$

$\int \theta\sin\theta\,d\theta = -\theta\cos\theta + \int\cos\theta\,d\theta = -\theta\cos\theta + \sin\theta$

so $\int_{-\pi}^{\pi}\theta\cos\theta\,d\theta = 0$ and $\int_{-\pi}^{\pi}\theta\sin\theta\,d\theta = 2\pi$]

So
$$[I] = \frac{m}{12}\begin{bmatrix} 6R^2+L^2 & 0 & 0 \\ 0 & 6R^2+L^2 & -\frac{6RL}{\pi} \\ 0 & -\frac{6RL}{\pi} & 12R^2 \end{bmatrix}$$

[50%]

(b) $R=0$: $I_{xx} = I_{yy} = \dfrac{mL^2}{12}$, $I_{zz}=0$ as in data book for straight rod

$L=0$: $I_{xx} = I_{yy} = \frac{1}{2}mR^2$, $I_{zz} = mR^2$ ✓

$L = 2R$:
$$[I] = \frac{mR^2}{6}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & -\frac{6}{\pi} \\ 0 & -\frac{6}{\pi} & 6 \end{bmatrix}$$

[10%]

(c) By parallel axis theorem $I = I_G + m\begin{bmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{bmatrix}$

Five coils with z being, say, $-2L$, $-L$, 0, L and $2L$

$x = y = 0$

$$\therefore \; I = \frac{5MR^2}{6}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & -6/\pi \\ 0 & -6/\pi & 6 \end{bmatrix} + m\begin{bmatrix} (4+1+0+1+4)L^2 & 0 & 0 \\ 0 & (4+1+0+1+4)L^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \frac{5MR^2}{6}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & -\frac{6}{\pi} \\ 0 & -\frac{6}{\pi} & 6 \end{bmatrix} + 10ML^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and with $L = 2R$ (gives $\frac{265}{6}MR^2$ on leading diagonal, etc). [20%]

(d) For the couple, $\underline{Q} = \dot{\underline{h}} = \Omega \underline{k} \times \underline{h}$

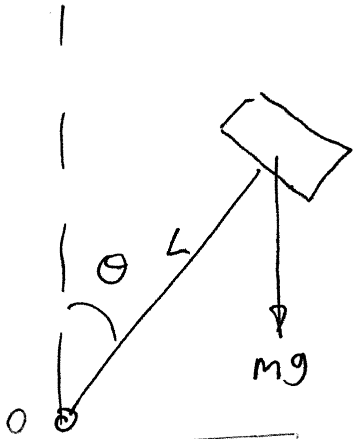
with $\underline{h} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ & I_{yy} & -I_{yz} \\ & & I_{zz} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix} \begin{matrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{matrix} = (-I_{xz}\,\underline{i} - I_{yz}\,\underline{j} + I_{zz}\,\underline{k})\Omega$

so $\underline{Q} = (-I_{xz}\,\underline{j} + I_{yz}\,\underline{i})\Omega^2$ but $I_{xz} = 0$

$I_{yz} = \frac{6}{\pi}\,\frac{5MR^2}{6}$

so $\underline{Q} = \frac{5MR^2\Omega^2}{\pi}\,\underline{i}$ [20%]

2/



Take the gyro equations relative to the Fixed point O.

Use gyro equation 2

$$Q_2 = A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1$$

with $Q_2 = mgL\sin\theta$

and $\dot{\Omega}_2 = 0$ in steady state

and $\Omega_3 = \dot{\phi}\cos\theta$

$\Omega_1 = -\dot{\phi}\sin\theta$

$$\therefore mgL\sin\theta = -(A\dot{\phi}\cos\theta - C\omega_3)\dot{\phi}\sin\theta$$

$$\therefore A\cos\theta\,\dot{\phi}^2 - C\omega_3\dot{\phi} + mgL = 0 \quad (1)$$

a/ Fast spin $\omega_3 \gg \dot{\phi}$ $\therefore$ ignore $\dot{\phi}^2$ term

$$\therefore \dot{\phi} \simeq \frac{mgL}{C\omega_3}$$

precession for fast spin at all θ [30%]

b/ For non-fast spin there is only a solution to (1) if the discriminant Δ is positive

ie $\Delta =$ "$b^2 - 4ac$" $= (C\omega_3)^2 - 4A\cos\theta\, mgL > 0$

$$\therefore \omega_3^2 > \frac{4A\cos\theta\, mgL}{C^2}$$

and for $\theta < \frac{\pi}{2}$, $\cos\theta$ is positive hence there is always a speed below which precession is impossible. [30%]

c/ for $\theta > \frac{\pi}{2}$ $\cos\theta$ is negative so Δ is always positive meaning that precession always occurs. [10%]

d/ The full solution to (1) is

$$\dot{\phi} = \frac{C\omega_3 \pm \sqrt{(C\omega_3)^2 - 4A\cos\theta\, mgL}}{2A\cos\theta}$$

and for θ close to π then $\cos\theta \sim -1$

so $$\dot{\phi} \simeq \frac{C\omega_3 \pm \sqrt{(C\omega_3)^2 + 4AMgL}}{2A}$$

and for $C\omega_3 \ll 4AMgL$

then $$\sqrt{(C\omega_3)^2 + 4AMgL} = \sqrt{4AMgL}\sqrt{1 + \frac{(C\omega_3)^2}{4AMgL}}$$

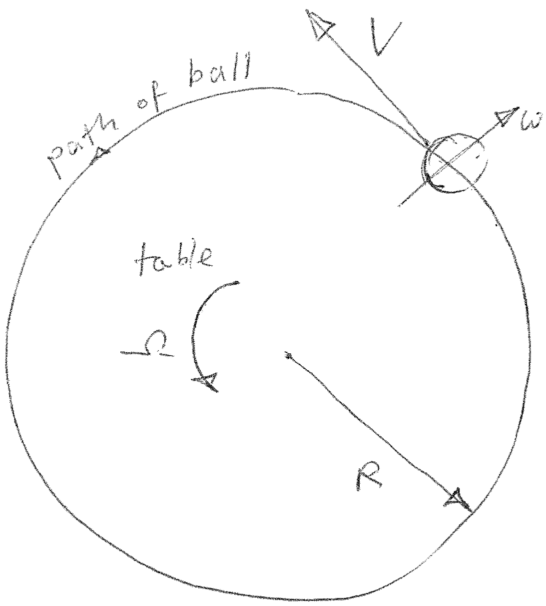
$$\simeq \sqrt{4AMgL}\left(1 + \frac{1}{2}\frac{(C\omega_3)^2}{4AMgL}\right)$$

so $$\dot{\phi} \simeq \pm\sqrt{\frac{MgL}{A}} + \frac{C\omega_3}{2A}$$

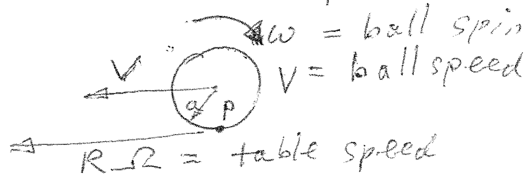
Note that $A \simeq ML^2$ if the rotor is treated as a point mass $\therefore$ $\dot{\phi} \simeq \pm\sqrt{\frac{g}{L}} + \frac{C\omega_3}{2A}$

↑ this is the frequency of a conical pendulum. The $\pm$ allows for the conical motion being "clockwise" or anticlockwise. the small spin ω_3 means that precession one way is slightly faster than the other way. [30%]

3.



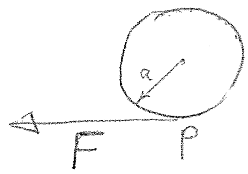
a/ Ball no slip at P



$\therefore \quad a\omega = R\Omega - V$

$\therefore \quad \omega = \dfrac{R\Omega - V}{a}$ [20%]

b/ Ball moves on a circular path so use $F = \dfrac{mV^2}{R}$ for the friction force at P



So couple $Q = Fa = \dfrac{mV^2 a}{R}$ [30%]

c/ Use "$Q = J\Omega_1\omega$" for gyroscopic effect with $J = \frac{2}{5}ma^2$ and $\Omega_1 = \frac{V}{R}$ = precession rate

$\therefore \quad \dfrac{mV^2 a}{R} = \dfrac{2}{5}ma^2 \, \dfrac{V}{R} \, \dfrac{R\Omega - V}{a}$

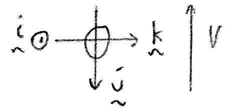
$\therefore \quad 5V = 2(R\Omega - V)$

$\therefore \quad \underline{\underline{V = \frac{2}{7}R\Omega}}$ [50%]

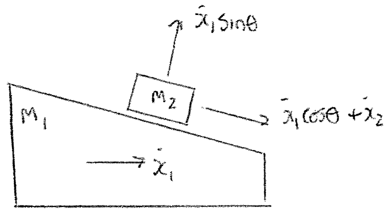
Note: $Q = J\Omega\omega$ was covered in Part IB Mechanics. To derive this result from the gyro equations, $\underset{\sim}{k}$ must be aligned along ω. The second gyro equation then gives:

$A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1 = Q_2$. Plan view

$A\dot{\Omega}_2 \to 0$, $(A\Omega_3 - C\omega_3) \to 0$, $Q_2 \to -\dfrac{mV^2 a}{R}$



4 a) Velocities :



Kinetic energy: $T = \frac{1}{2}M_1\dot{x}_1^2 + \frac{1}{2}M_2\{(\dot{x}_1\sin\theta)^2 + (\dot{x}_1\cos\theta + \dot{x}_2)^2\}$

$$\Rightarrow T = \frac{1}{2}M_1\dot{x}_1^2 + \frac{1}{2}M_2(\dot{x}_1^2 + \dot{x}_2^2 + 2\dot{x}_1\dot{x}_2\cos\theta)$$

Potential energy : $U = \frac{1}{2}kx_2^2 + M_2g(-x_2\sin\theta)$

Generalised Force : $\delta W = P\delta(x_1\cos\theta + x_2) = (P\cos\theta)\delta x_1 + P\delta x_2 \Rightarrow Q_1 = P\cos\theta,\ Q_2 = P$

Lagrange for x_1 : $\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_1}\right] - \frac{\partial T}{\partial x_1} + \frac{\partial U}{\partial x_1} = Q_1$

$$\Rightarrow M_1\ddot{x}_1 + M_1\ddot{x}_2 + M_2\ddot{x}_2\cos\theta = P\cos\theta$$

Lagrange for x_2 : $\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_2}\right] - \frac{\partial T}{\partial x_2} + \frac{\partial U}{\partial x_2} = Q_2$

$$\Rightarrow M_2\ddot{x}_2 + M_2\ddot{x}_1\cos\theta + kx_2 - M_2g\sin\theta = P$$

In matrix form:

$$\begin{pmatrix} M_1+M_2 & M_2\cos\theta \\ M_2\cos\theta & M_2 \end{pmatrix}\begin{pmatrix}\ddot{x}_1 \\ \ddot{x}_2\end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & k\end{pmatrix}\begin{pmatrix}x_1 \\ x_2\end{pmatrix} = \begin{pmatrix}P\cos\theta \\ P + M_2g\sin\theta\end{pmatrix}$$

[40%]

b) For free vibration $|-\omega^2 M + k| = 0 \Rightarrow \begin{vmatrix} -(M_1+M_2)\omega^2 & -M_2\cos\theta\,\omega^2 \\ -M_2\cos\theta\,\omega^2 & -M_2\omega^2 + k\end{vmatrix} = 0$

$$\Rightarrow \omega^2\{-(M_1+M_2)(k - M_2\omega^2) - M_2^2\omega^2\cos^2\theta\} = 0$$

$$\omega^2\{-(M_1+M_2)k + M_2\omega^2(M_1 + M_2 - M_2\cos^2\theta)\} = 0$$

$$\Rightarrow \omega^2 = 0 \quad \text{or} \quad \omega^2 = \frac{k(M_1+M_2)}{M_2(M_1 + M_2\sin^2\theta)}$$

for mode shapes $\begin{pmatrix}-M_2\cos\theta\,\omega^2 & -M_2\omega^2 + k\end{pmatrix}\begin{pmatrix}x_1 \\ x_2\end{pmatrix} = 0 \xrightarrow{\omega = 0} \begin{pmatrix}x_1 \\ x_2\end{pmatrix} = \begin{pmatrix}1 \\ 0\end{pmatrix}$ rigid body motion

$$\begin{pmatrix}-(M_1+M_2)\omega^2 & -M_2\cos\theta\,\omega^2\end{pmatrix}\begin{pmatrix}x_1 \\ x_2\end{pmatrix} = 0 \xrightarrow{\omega = \omega_2} \begin{pmatrix}x_1 \\ x_2\end{pmatrix} = \begin{pmatrix}\frac{-M_2\cos\theta}{M_1+M_2} \\ 1\end{pmatrix}$$

[60%]

c) Generalised momentum $= \frac{\partial T}{\partial \dot{q}_i}$

For x_1 : $\frac{\partial T}{\partial \dot{x}_1} = (M_1+M_2)\dot{x}_1 + M_2\dot{x}_2\cos\theta$ $\longrightarrow$ zero (constant) for both modes 1 and 2 since no other forces in horizontal direction

For x_2 : $\frac{\partial T}{\partial \dot{x}_2} = M_2\dot{x}_2 + M_2\dot{x}_1\cos\theta$ $\longrightarrow$ not zero for mode 2, spring provides a force. [20%]

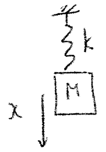
5 a) Lagrange's equation:

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i$$

gravity can be included in one of these terms, but not both.

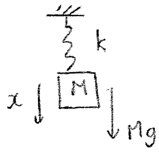
(i) Mass-spring system:

Gravity in potential



$T = \frac{1}{2}M\dot{x}^2$, $U = -Mgx$, $Q = 0$ $\longrightarrow$ Lagrange $\underline{M\ddot{x} + kx = Mg}$

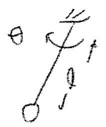
Gravity in generalised Force



$T = \frac{1}{2}M\dot{x}^2$, $U = 0$, $\delta W = Mg\,\delta x$ $\Rightarrow Q = Mg$ $\longrightarrow$ Lagrange $\underline{M\ddot{x} + kx = Mg}$ [15%]

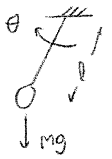
(ii) Pendulum system

Gravity in potential



$T = \frac{1}{2}Ml^2\dot{\theta}^2$, $U = Mgl(1-\cos\theta)$, $Q = 0$ $\longrightarrow$ Lagrange $\underline{Ml^2\ddot{\theta} + Mgl\sin\theta = 0}$

Gravity in generalised Force



$T = \frac{1}{2}Ml^2\dot{\theta}^2$

$U = 0$

$\delta W = -Mg\,\delta(1-\cos\theta)l$

$= -mg\sin\theta.\delta\theta.l$

$\Rightarrow Q = -Mgl\sin\theta$

$\longrightarrow$ Lagrange $\underline{Ml^2\ddot{\theta} + Mgl\sin\theta = 0}$ [20%]

b) $T = \frac{1}{2}M\dot{x}^2$, $U = \frac{1}{2}kx^2$, $R = \frac{1}{2}\lambda\dot{x}^2$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}}\right] - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} + \frac{\partial R}{\partial \dot{x}} = 0 \quad \Rightarrow \quad \underline{M\ddot{x} + \lambda\dot{x} + kx = 0}$$ ✓ [15%]

c) (i) $T = \frac{1}{2}\dot{\underset{\sim}{q}}^T M \dot{\underset{\sim}{q}} = \frac{1}{2}\sum_i\sum_j M_{ij}\dot{q}_i\dot{q}_j$

$U = \frac{1}{2}\underset{\sim}{q}^T k \underset{\sim}{q} = \frac{1}{2}\sum_i\sum_j k_{ij} q_i q_j$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i$$

$\Rightarrow \sum_j M_{ij}\ddot{q}_j + \sum_j k_{ij}q_j = Q_i$ for all i

$\Rightarrow M\ddot{\underset{\sim}{q}} + k\underset{\sim}{q} = \underset{\sim}{Q}$ [20%]

(ii) By direct analogy with the above, if $F = \frac{1}{2}\dot{\underset{\sim}{q}}^T C \dot{\underset{\sim}{q}} = \frac{1}{2}\sum_i\sum_j C_{ij}\dot{q}_i\dot{q}_j$

$$\Rightarrow \frac{\partial F}{\partial \dot{q}_i} = \sum_j C_{ij}\dot{q}_j \quad \text{(if } C \text{ is symmetric)}$$

$$\Rightarrow \quad C\dot{\underset{\sim}{q}} \text{ in equation of Motion}$$

[20%]

(iii) The term $\frac{\partial F}{\partial \dot{q}_j}$ is <u>linear</u> in $\dot{\underset{\sim}{q}}$ $\Rightarrow$ F is <u>second order</u> in $\dot{\underset{\sim}{q}}$

↓

Taylor series expression $F \equiv \sum_i\sum_j \underbrace{\frac{\partial^2 F}{\partial \dot{q}_i \partial \dot{q}_j}}_{\text{symmetric}} \dot{q}_i\dot{q}_j$

$\Rightarrow$ only symmetric damping matrices can be produced in this way.

[10%]

3C5 Dynamics: Answers to Tripos Paper 2010

1. (a) $\mathbf{I}=\frac{m}{12}\begin{pmatrix} 6R^2+L^2 & 0 & 0 \\ 0 & 6R^2+L^2 & -6RL/\pi \\ 0 & -6RL/\pi & 12R^2 \end{pmatrix}$.

(b) $\mathbf{I}=\frac{5mR^2}{6}\begin{pmatrix} 53 & 0 & 0 \\ 0 & 53 & -6/\pi \\ 0 & -6/\pi & 6 \end{pmatrix}$.

(c) $\mathbf{Q}=(5mR^2\Omega^2/\pi)\mathbf{i}$.

2. (a) $\dot{\phi}=mgL/(C\omega_3)$.

(b) Stable for $\omega_3^2>(4AmgL/C^2)\cos\theta$.

(c) Always stable because $\cos\theta<0$.

(d) For $A\approx mL^2$ the two solutions are $\dot{\phi}=\frac{C\omega_3}{2A}\pm\sqrt{\frac{g}{L}}$.

3. (b) $Q=maV^2/R$ in the direction of the velocity V.

(c) $\alpha=2/7$.

4. (a) $\begin{pmatrix} m_1+m_2 & m_2\cos\theta \\ m_2\cos\theta & m_2 \end{pmatrix}\begin{pmatrix}\ddot{x}_1 \\ \ddot{x}_2\end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 0 & k\end{pmatrix}\begin{pmatrix} x_1 \\ x_2\end{pmatrix}=\begin{pmatrix} P\cos\theta \\ P+m_2 g\sin\theta\end{pmatrix}$.

(b) $\omega_1=0,\ \mathbf{u}_1=\begin{pmatrix}1\\0\end{pmatrix};\ \omega_2^2=\frac{k(m_1+m_2)}{m_2(m_1+m_2\sin^2\theta)},\ \mathbf{u}_2=\begin{pmatrix}\frac{-m_2\cos\theta}{m_1+m_2} \\ 1\end{pmatrix}$.

(c) $p_1=(m_1+m_2)\dot{x}_1+m_2\dot{x}_2\cos\theta$ (conserved)
$p_2=m_2\dot{x}_2+m_2\dot{x}_1\cos\theta$ (not conserved).

5. (b) $M\ddot{x}+\lambda\dot{x}+Kx=Mg$.

(c) (i) $T=(1/2)\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{q}}$, $V=(1/2)\mathbf{q}^{\mathrm{T}}\mathbf{K}\mathbf{q}$.

(ii) $F=(1/2)\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{C}\dot{\mathbf{q}}$, (iii) not possible.