# Part IIa Paper 3C6, 2010

(a) If spring is infinitely stiff, the base is fixed and each beam is constrained to have zero displacement and zero slope at its attachment to A-B. So follow the usual method for a cantilever

$$EI\frac{d^4u}{dx^4} = m\omega^2 u$$

where $w(x,t) = u(x)e^{i\omega t}$

So $u = K_1\cos\alpha x + K_2\sin\alpha x + K_3\cosh\alpha x + K_4\sinh\alpha x$

with $\alpha^4 = m\omega^2/EI$

Beam is clamped at $x=0$, free at $x=L$

So $\begin{cases} u(0)=0 & \rightarrow \quad K_1 + K_3 = 0 \\ u'(0) = 0 & \rightarrow \quad K_2 + K_4 = 0 \end{cases}$

$u''(L) = 0 \rightarrow K_1(-\cos\alpha L - \cosh\alpha L) + K_2(-\sin\alpha L - \sinh\alpha L) = 0$

$u'''(L) = 0 \rightarrow K_1(\sin\alpha L - \sinh\alpha L) + K_2(-\cos\alpha L - \cosh\alpha L) = 0$

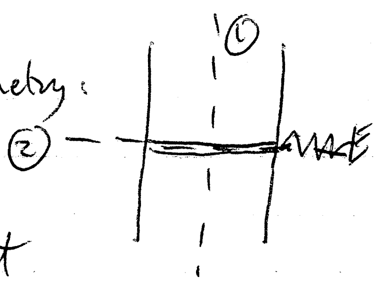
So determinant $|\because| = 0$

$\rightarrow (\cos\alpha L + \cosh\alpha L)^2 = -(\sin\alpha L - \sinh\alpha L)(\sin\alpha L + \sinh\alpha L)$

$\therefore \cos^2\alpha L + 2\cos\alpha L\cosh\alpha L + \cosh^2\alpha L = \sinh^2\alpha L - \sin^2\alpha L$

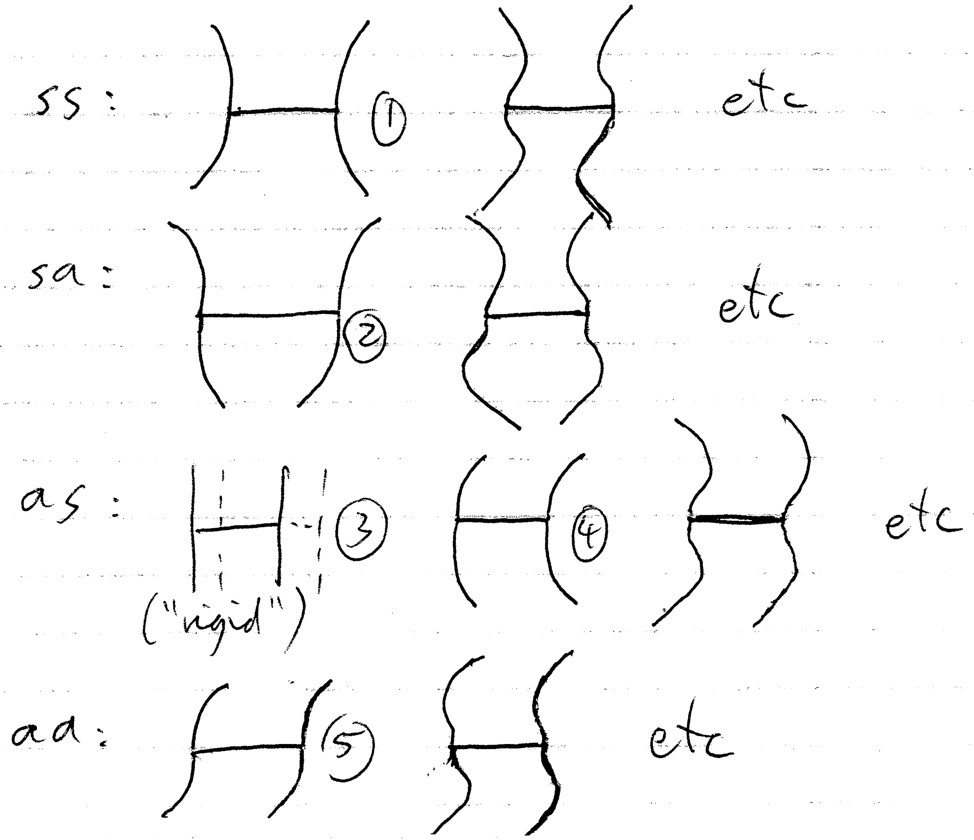
$\therefore \cos\alpha L\cosh\alpha L = -1$

(b) The system has two planes of symmetry:




Each vibration mode must be either symmetric or antisymmetric with respect to each plane, so there are four groups of modes sym/sym, sym/anti, anti/sym, anti/anti.
Sketches of the first few in each group:

1b(cont)




1(c) With the spring removed mode 3 has zero frequency. Modes 1, 2 and 5 are all unchanged because they exert no net force on AB so it stays still. Mode 4 is the only one of this set which changes, together with the corresponding as mode of every higher group of 4.

To find this frequency, consider a single beam which is still free at $x = L$, but which has
$$\begin{cases} u' = 0 & \text{(no rotation)} \\ u''' = 0 & \text{(no shear force)} \end{cases}$$
at $x = 0$.

So repeat calculation from (b):

$u'(0) = 0 \rightarrow K_2 + K_4 = 0$

$u'''(0) = 0 \rightarrow -K_2 + K_4 = 0$

$\therefore K_2 = K_4 = 0$

$u''(L) = 0 \rightarrow -K_1 \cos \alpha L + K_3 \cosh \alpha L = 0$

$u'''(L) = 0 \rightarrow K_1 \sin \alpha L + K_3 \sinh \alpha L = 0$

$\therefore \cos \alpha L \sinh \alpha L + \sin \alpha L \cosh \alpha L = 0$

$\therefore \tan \alpha L = - \tanh \alpha L$

2 (a) Governing equation from data sheet $\rho \frac{\partial^2 w}{\partial t^2} - E \frac{\partial^2 w}{\partial x^2} = 0$

Boundary conditions: $\begin{cases} w = 0 \text{ at } x = 0 \\ \frac{\partial w}{\partial x} = 0 \text{ at } x = L \text{ (no strain)} \end{cases}$

So let $w = u(x) e^{i\omega t}$ for modes:

Then $u'' = -\frac{\omega^2}{c^2} u$, $c^2 = \frac{E}{\rho}$

General solution $u = K_1 \sin \omega x/c + K_2 \cos \omega x/c$

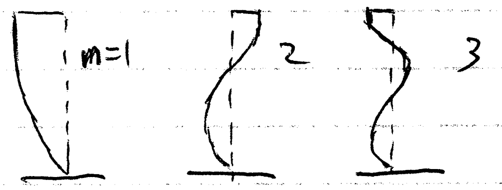
$u(0) = 0 \rightarrow K_2 = 0$

$u'(L) = 0 \rightarrow K_1 \frac{\omega}{c} \cos \frac{\omega L}{c} = 0$

So natural frequencies where $\cos \frac{\omega L}{c} = 0$, so $\frac{\omega L}{c} = (n - \frac{1}{2})\pi$ $n = 1, 2, 3 \ldots$

Corresponding mode shape $u_n(x) = K_1 \sin \frac{(n-\frac{1}{2})\pi x}{L}$.

Sketches:




(b) Static equilibrium under end force must have $w = \lambda x$ for some value $\lambda$.

Condition at $x = L$ is $F = EA \frac{\partial w}{\partial x} = \lambda EA$, $\therefore \lambda = \frac{F}{EA}$

Now at $t = 0$ force at $x = L$ jumps <u>down</u> from F to 0.
This is a <u>negative</u> step by $-F$, superposed on the static solution at $t = 0$.

$\therefore$ Transient response $w(x,t) = \frac{Fx}{EA} - F \times (\text{step response})$

Use Data Sheet result P3 #10, but first need to normalise modes.

2 cont

We require $\int_0^L K_1^2 \sin^2 \frac{w_n x}{c} \cdot \rho A \, dx = 1$

But $\int_0^L \sin^2 \frac{(n-\frac{1}{2})\pi x}{L} \, dx = \frac{L}{2}$, so need $K_1^2 = \frac{2}{\rho A L}$

So transient response is $w(x,t) = \frac{Fx}{EA} - F \sum_{n=1}^{\infty} \frac{2}{\rho A L} \frac{u_n(x) u_n(L)}{w_n^2} (1 - \cos w_n t)$

$$= \frac{Fx}{EA} - \frac{2F}{\rho A L} \sum_{n=1}^{\infty} \frac{\sin (n-\frac{1}{2})\pi \, \sin \frac{(n-\frac{1}{2})\pi x}{L}}{c^2 \left(\frac{(n-\frac{1}{2})\pi}{L}\right)^2} \left[1 - \cos \frac{(n-\frac{1}{2})\pi c t}{L}\right]$$

$$= \frac{F}{EA} \left\{ x - \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n-\frac{1}{2})^2} \sin \frac{(n-\frac{1}{2})\pi x}{L} \left[1 - \cos \frac{(n-\frac{1}{2})\pi c t}{L}\right] \right\} \quad (1)$$

But using the given Fourier series, the first two terms cancel, leaving

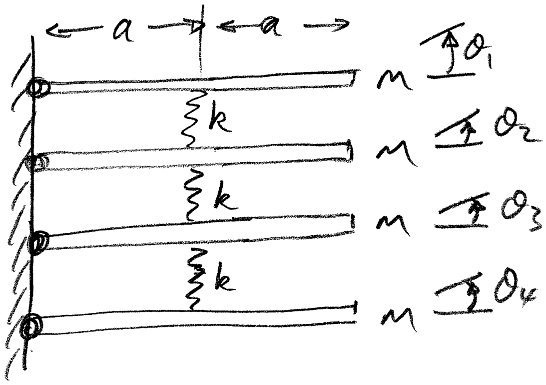
$$w(x,t) = \frac{2LF}{EA\pi^2} \sum_{n=1}^{\infty} \sin \frac{(n-\frac{1}{2})\pi x}{L} \cos \frac{(n-\frac{1}{2})\pi c t}{L} \frac{(-1)^{n+1}}{(n-\frac{1}{2})^2}$$

(c) This cancellation <u>had</u> to happen in order that when small damping is added we get $w \to 0$ as $t \to \infty$.

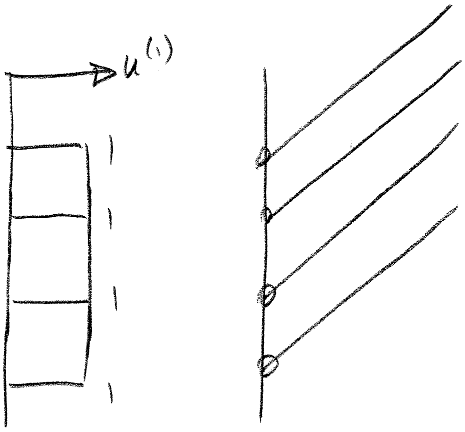
Data Sheet formula says that the factor $[1 - \cos w_n t]$ in (1) is replaced by $[1 - \cos w_n t \, e^{-w_n \zeta_n t}]$

The exponential factor $\to 0$ as $t \to \infty$, leaving the two terms in (1) which cancelled.

3.

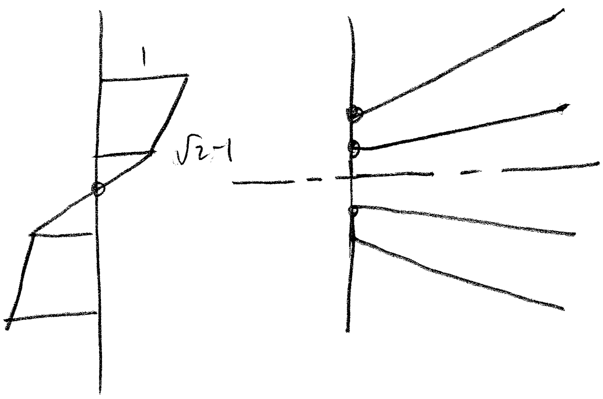



$$T_1 = \tfrac{1}{2} I_G \dot{\theta}_1^2 + \tfrac{1}{2} m (a\dot{\theta}_1)^2$$
$$= \tfrac{1}{2}\left[\tfrac{4}{3} m a^2 \dot{\theta}_1^2\right]$$
$$\Rightarrow T = \tfrac{1}{2}\left[\tfrac{4}{3} m a^2 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2 + \dot{\theta}_4^2\right)\right]$$
$$V = \tfrac{1}{2} k \left[a\theta_1 - a\theta_2\right)^2 + \tfrac{1}{2} k (a\theta_2 - a\theta_3)^2 + \tfrac{1}{2} k (a\theta_3 - a\theta_4)^2]$$
$$\therefore\ V = \tfrac{1}{2} k a^2 \left[\theta_1^2 + 2\theta_2^2 + 2\theta_3^2 + \theta_4^2 - 2\theta_1\theta_2 - 2\theta_2\theta_3 - 2\theta_3\theta_4\right]$$



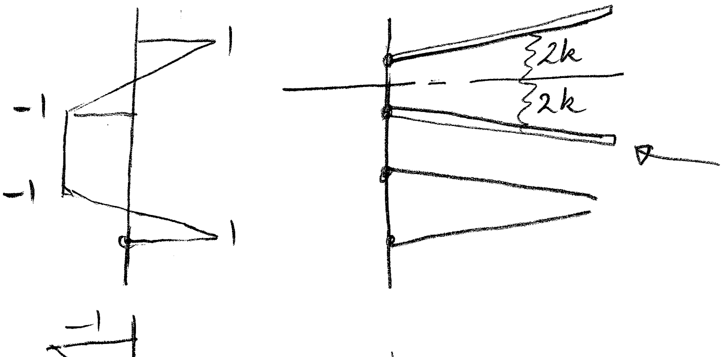

Rigid body mode $\omega_1 = 0$.

$u^{(1)} = [1\ 1\ 1\ 1]^T$



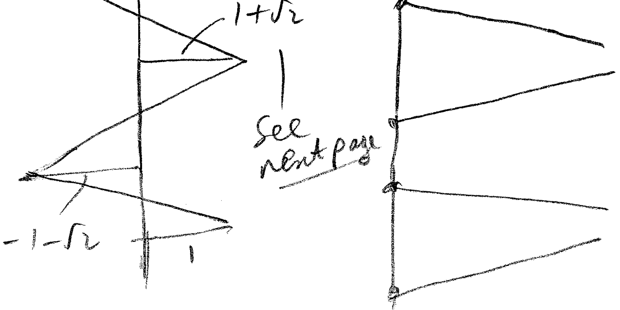

$u^{(2)} = [1\ \ \alpha\ \ -\alpha\ \ -1]$

$\omega_2$




$u^{(3)} = [1\ \ -1\ \ -1\ \ 1]$

$$\tfrac{4}{3} m a^2 \ddot{\theta} + 2k(a\theta) = 0$$
$$\ddot{\theta} + \tfrac{3}{2}\tfrac{k}{m}\theta = 0$$
$$\omega_3 = \sqrt{\tfrac{3k}{2m}}$$




$u^{(4)} = [-1\ \ \alpha\ \ -\alpha\ \ 1]$

Same form as $u^{(2)}$

$\omega_4$

3 (cont)

Rayleigh $\quad \omega^2 = \dfrac{V_{max}}{T^*} \quad$ with $u = [1 \quad \alpha \quad -\alpha \quad -1]$

$$\omega^2 = \tfrac{1}{2} k a^2 \left[ (1-\alpha)^2 + (\alpha + \alpha)^2 + (-\alpha+1)^2 \right]$$

$$= \tfrac{1}{2} k a^2 \left[ 2 - 4\alpha + 6\alpha^2 \right]$$

$$T = \tfrac{1}{2} \tfrac{4}{3} m a^2 \left[ 1 + \alpha^2 + \alpha^2 + 1 \right] = \tfrac{1}{2} \tfrac{4}{3} m a^2 \left[ 2 + 2\alpha^2 \right]$$

$$\therefore \quad \omega^2 = \frac{\cancel{\tfrac{1}{2}} k \cancel{a^2} \left[ 2 - 4\alpha + 6\alpha^2 \right]}{\cancel{\tfrac{1}{2}} \tfrac{4}{3} m \cancel{a^2} \left[ 2 + 2\alpha^2 \right]} = \frac{3k}{4m} \frac{\left[ 2 - 4\alpha + 6\alpha^2 \right]}{\left[ 2 + 2\alpha^2 \right]}$$

<u>To find $\alpha$ :</u>

$$\frac{d\omega^2}{d\alpha} = 0 : \quad \frac{3k}{4m} \left[ \frac{(2+2\alpha^2)(-4+12\alpha) - (2-4\alpha+6\alpha^2)(4\alpha)}{(2+2\alpha^2)^2} \right] = 0$$

$$\Rightarrow \quad -8 + 24\alpha - 8\alpha^2 + \cancel{24\alpha^3} - 8\alpha + 16\alpha^2 - \cancel{24\alpha^3} = 0$$

$$\Rightarrow \quad 8\alpha^2 + 16\alpha - 8 = 0$$

$$\Rightarrow \quad \alpha^2 + 2\alpha - 1 = 0 \qquad \Rightarrow \quad \alpha = -1 \pm \sqrt{2}$$

for $\alpha = -1 - \sqrt{2}$, $= -2.414$

$$\omega^2 = \frac{3k}{4m} \left[ \frac{2 - 4(-1-\sqrt{2}) + 6(-1-\sqrt{2})^2}{2 + 2(-1-\sqrt{2})^2} \right]$$

$$= \frac{3k}{4m} \left[ \frac{6 + 4\sqrt{2}}{2 + \sqrt{2}} \right] = 2.561 \frac{k}{m}$$

$$\Rightarrow \quad \omega_4 = 1.60 \sqrt{\frac{k}{m}}$$

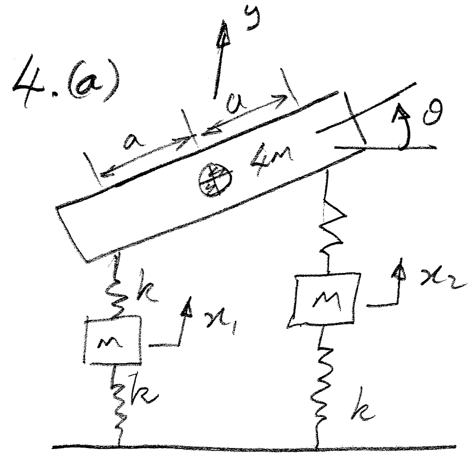
for $\alpha = -1 + \sqrt{2}$, $= 0.414$

$$\omega^2 = \frac{3k}{4m} \left[ \frac{2 - 4(-1+\sqrt{2}) + 6(-1+\sqrt{2})^2}{2 + 2(-1+\sqrt{2})^2} \right]$$

$$= \frac{3k}{4m} \left[ \frac{6 - 4\sqrt{2}}{2 - \sqrt{2}} \right] = 0.4393 \frac{k}{m}$$

$$\omega_2 = 0.663 \sqrt{\frac{k}{m}}$$

4.(a)




$$T = \tfrac{1}{2} m \dot{x}_1^2 + \tfrac{1}{2} m \dot{x}_2^2 + \tfrac{1}{2} 4m \dot{y}^2 + \tfrac{1}{2} m a^2 \dot{\theta}^2$$

$$[M] = m \begin{array}{c} \begin{array}{cccc} \dot{x}_1 & \dot{x}_2 & \dot{y} & \dot{\theta} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & a^2 \end{bmatrix} \end{array}$$

$$V = \tfrac{1}{2} k x_1^2 + \tfrac{1}{2} k x_2^2 + \tfrac{1}{2} k (y - a\theta - x_1)^2 + \tfrac{1}{2} k (y + a\theta - x_2)^2$$

$$= \tfrac{1}{2} k \left(2x_1^2 + 2x_2^2 + 2y^2 + 2a^2\theta^2 - 2x_1 y - 2x_2 y + 2a x_1 \theta - 2a x_2 \theta\right)$$

Hence by inspection of the quadratic form:

$$V = \tfrac{1}{2} [x_1 \; x_2 \; y \; \theta]\, k \underbrace{\begin{bmatrix} 2 & 0 & -1 & a \\ 0 & 2 & -1 & -a \\ -1 & -1 & 2 & 0 \\ a & -a & 0 & 2a^2 \end{bmatrix}}_{[K]} \begin{Bmatrix} x_1 \\ x_2 \\ y \\ \theta \end{Bmatrix}$$

(or do this by Lagrange)

(b) For the natural modes, $\left([K] - \omega^2 [M]\right) \underset{\sim}{u} = 0$

for $\underline{u} = [1 \;\; 1 \;\; \alpha \;\; 0]^T$ this is

$$\begin{bmatrix} 2k - \omega^2 m & 0 & -k & ak \\ 0 & 2k - \omega^2 m & -k & -ak \\ -k & -k & 2k - 4\omega^2 m & 0 \\ ak & -ak & 0 & a^2(2k - \omega^2 m) \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ \alpha \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (1)$$

First row of (1) is: $2k - \omega^2 m - k\alpha = 0 \Rightarrow (2 - \alpha) k = \omega^2 m$ — (2)

Third row of (1) is: $-2k + (2k - 4\omega^2 m)\alpha = 0 \Rightarrow (\alpha - 1) k = 2\alpha \omega^2 m$ — (3)

$$\frac{(2)}{(3)} \Rightarrow \frac{2 - \alpha}{\alpha - 1} = \frac{1}{2\alpha} \Rightarrow 2\alpha^2 - 3\alpha - 1 = 0$$

$$\left.\alpha = \frac{3}{4} \pm \frac{\sqrt{17}}{4} = \begin{array}{l} 1.7808 \\ -0.2803 \end{array}\right\} \quad (2) \Rightarrow \begin{array}{l} \omega^2 = 0.219\, k/m \\ \& \; \omega^2 = 2.281\, k/m \end{array}$$

4(b) Similarly for $\underset{\sim}{u} = [1 \ \ -1 \ \ 0 \ \ \beta]^T$

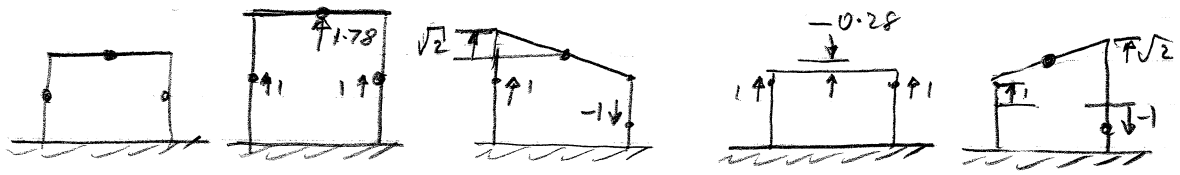
1st row of ①: $2k - \omega^2 m + ak\beta = 0 \Rightarrow (2 + a\beta)k = \omega^2 m$ — ④

4th row of ①: $2ak + a^2(2k - \omega^2 m)\beta = 0 \Rightarrow 2k(1 + a\beta) = \omega^2 m a\beta$ — ⑤

$$\frac{④}{⑤} \Rightarrow \frac{2 + a\beta}{2(1 + a\beta)} = \frac{1}{a\beta} \Rightarrow \cancel{2a\beta} + (a\beta)^2 = 2 + \cancel{2a\beta} \Rightarrow \underline{\underline{a\beta = \pm\sqrt{2}}}$$

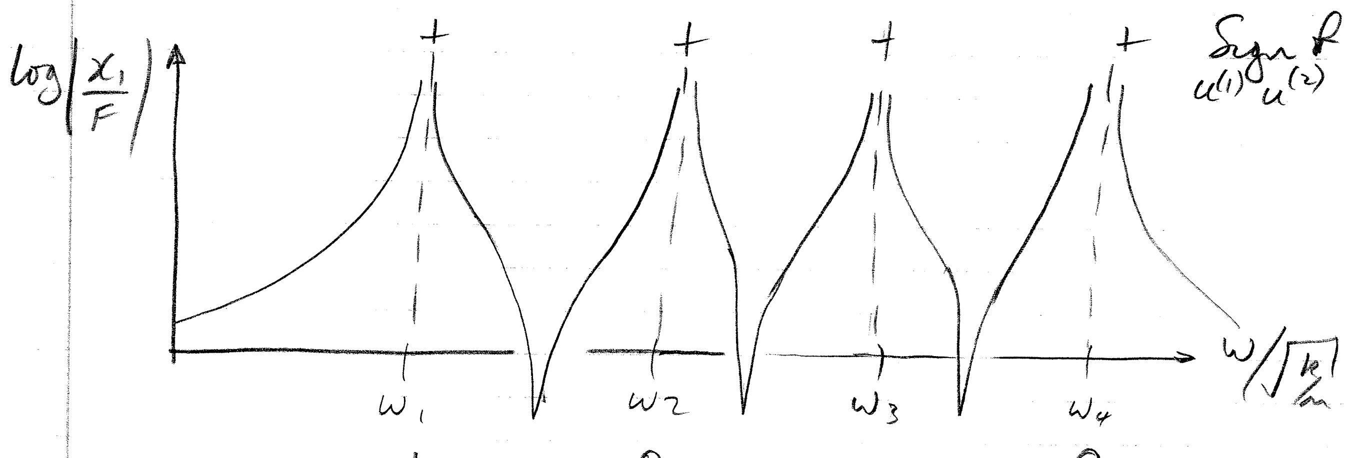
So ④ gives $\omega^2 = (2 \pm \sqrt{2})\frac{k}{m} = \underline{\begin{matrix} 0.586\,k/m \\ 3.414\,k/m \end{matrix}}$ (independent of the value of 'a')

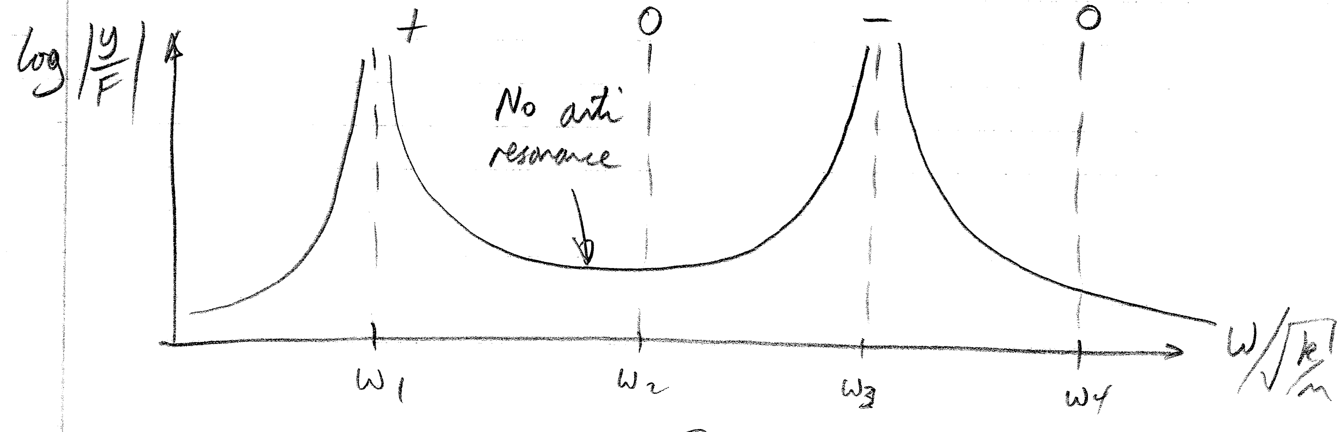
Mode shapes:




$\alpha = 1.7808$, $\omega^2 = 0.219\frac{k}{m}$ | $a\beta = -\sqrt{2}$, $\omega^2 = 0.586\,k/m$ | $\alpha = -0.280$, $\omega^2 = 2.281\,k/m$ | $a\beta = \sqrt{2}$, $\omega^2 = 3.414\frac{k}{m}$

(c) $$\frac{x_j}{F_k} = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega^2 - \omega_n^2}$$



ENGINEERING TRIPOS PART IIA

Module 3C6 Examination, 2010

Answers

2 (a) u(0,t)=0, u'(L,t)=0

(b) $u(x,t) = \dfrac{2LF}{EA\pi^2}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{(n-1/2)^2}\sin\left(\dfrac{(n-1/2)\pi x}{L}\right)\cos\left(\dfrac{(n-1/2)\pi ct}{L}\right).$

(c) $u(x,t) = \dfrac{2LF}{EA\pi^2}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{(n-1/2)^2}\sin\left(\dfrac{(n-1/2)\pi x}{L}\right)\cos\left(\dfrac{(n-1/2)\pi ct}{L}\right)e^{-\zeta_n \omega_n t}$

3 (a) $T = \dfrac{2}{3}ma^2\left(\dot\theta_1^2 + \dot\theta_2^2 + \dot\theta_3^2 + \dot\theta_4^2\right),$

$V = \dfrac{1}{2}ka^2\left[(\theta_1-\theta_2)^2 + (\theta_2-\theta_3)^2 + (\theta_3-\theta_4)^2\right]$

(c) $[1\ 1\ 1\ 1]^T$, $\omega_1 = 0$, $\omega_3 = \sqrt{\dfrac{3k}{2m}}$

(d) $\alpha = -1 \pm \sqrt{2}$, $\omega_2 = 0.663\sqrt{\dfrac{k}{m}}$, $\omega_4 = 1.60\sqrt{\dfrac{k}{m}}$, all frequencies are exact.

4 (a) $T = \dfrac{1}{2}m\dot x_1^2 + \dfrac{1}{2}m\dot x_2^2 + 2m\dot y^2 + \dfrac{1}{2}ma^2\dot\theta^2$, $[M] = m\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&4&0\\0&0&0&a^2\end{bmatrix}.$

$V = \dfrac{1}{2}k\left[x_1^2 + x_2^2 + (y-a\theta-x_1)^2 + (y+a\theta-x_2)^2\right]$

(b) $\alpha = \dfrac{3}{4} \pm \dfrac{\sqrt{17}}{4}$, $\omega^2 = 0.219\dfrac{k}{m}, 2.281\dfrac{k}{m}$; $\beta = \pm\dfrac{\sqrt{2}}{a}$, $\omega^2 = \left(2\pm\sqrt{2}\right)\dfrac{k}{m}$