ENGINEERING TRIPOS IIA 2010

3C7 MECHANICS OF SOLIDS

2010 PART IIA 3C7 MECHANICS OF SOLIDS
DR S D GUEST

1(a)

$$\frac{\partial \phi}{\partial x} = w\left[\frac{-x}{2} - \frac{6xy}{4d} + \frac{1}{20d^3}\cdot 40xy^3\right]$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = w\left[-\frac{1}{2} - \frac{6y}{4d} + \frac{2y^3}{d^3}\right]$$

$$\frac{\partial \phi}{\partial y} = w\left[-\frac{3x^2}{4d} + \frac{1}{20d^3}\left(6d^2y^2 - 15\ell^2y^2 + 60x^2y^2 - 20y^4\right)\right]$$

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = \frac{w}{10d^3}\left[6d^2y - 15\ell^2 y + 60x^2y - 40y^3\right]$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = w\left[-\frac{6xy^2}{d^3} + \frac{3x}{2d}\right]$$

Check compatibility

$$\frac{\partial^4 \phi}{\partial x^4} = 0$$

$$\frac{\partial^4 \phi}{\partial x^2 \partial y^2} = \frac{12wy}{d^3}$$

$$\frac{\partial^4 \phi}{\partial y^4} = -\frac{24wy}{d^3}$$

$$\nabla^4 \phi = \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \quad \checkmark$$

Not required for question

---

1 (a) cont. Check boundary conditions on stress at $y = \pm d/2$

At $y = +\frac{d}{2}$

$$\sigma_{yy} = w\left[-\frac{1}{2} - \frac{6}{8} + \frac{2}{8}\right] = -w \quad ✓$$

$$\sigma_{xy} = \frac{wx}{d}\left[\frac{6}{4} - \frac{3}{2}\right] = 0 \quad ✓$$

At $y = -\frac{d}{2}$

$$\sigma_{yy} = w\left[-\frac{1}{2} + \frac{6}{8} - \frac{2}{8}\right] = 0 \quad ✓$$

$$\sigma_{xy} = \frac{wx}{d}\left[\frac{6}{4} - \frac{3}{2}\right] = 0 \quad ✓$$

Conditions satisfied ~~on~~ top & bottom face.

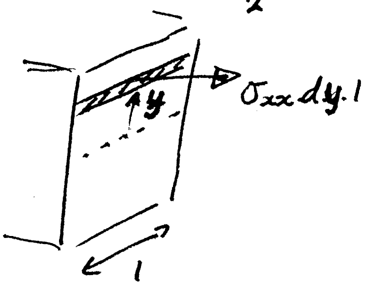
Only one candidate checked S, M & T

(b) We will, at best, be able to satisfy stress resultants at $x = \pm \ell/2$: moment $= 0$, tension $= 0$, shear force $= \frac{w\ell}{2}$ up.

Tension

$$T = \int_{-\frac{d}{2}}^{\frac{d}{2}} \sigma_{xx} \cdot 1 \cdot dy$$

take unit depth



$= 0$ for all x, as σ_{xx} is an odd function in y

Moment

$$M = \int_{-d/2}^{d/2} \sigma_{xx} \cdot y \cdot dy$$

At $x = \pm\frac{\ell}{2}$, $\sigma_{xx} = \frac{w}{10d^3}\left(6d^2y - 40y^3\right)$

1(b) cont.

At $x = \pm \ell/2$

$$M = \frac{w}{10d^3}\int_{-d/2}^{d/2} \left(6d^2y^2 - 40y^4\right) dy$$

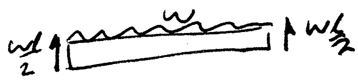
$$= \frac{m}{10d^3}\left[2d^2y^3 - 8y^5\right]_{-d/2}^{d/2} = 0 \quad \checkmark$$

Shear force: from overall equilibrium, this must hold, but still check:

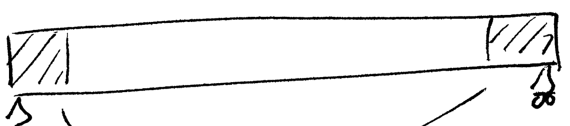
$$S = \int_{-d/2}^{d/2} \sigma_{xy}\, dy = \frac{wx}{d}\int_{-d/2}^{d/2}\left(\frac{3}{2} - \frac{6y^2}{d^2}\right) dy$$

$$= \frac{wx}{d}\left[\frac{3y}{2} - \frac{2y^3}{d^3}\right]_{-d/2}^{d/2} = wx$$

at $x = \frac{\ell}{2}$, $S = \frac{w\ell}{2}$ ✓ ; at $x = -\frac{\ell}{2}$, $S = -\frac{w\ell}{2}$ ✓



Thus we have the stresses in a simply-supported beam — but for a particular set of stresses at the supports which may not exactly match the particular supports shown. By Saint Venant's principle, we would expect the stresses to be accurate away from the supports (order of d away), but not around supports



stresses not accurate here.

Most candidates did not mention Saint Venant

1. Cont.

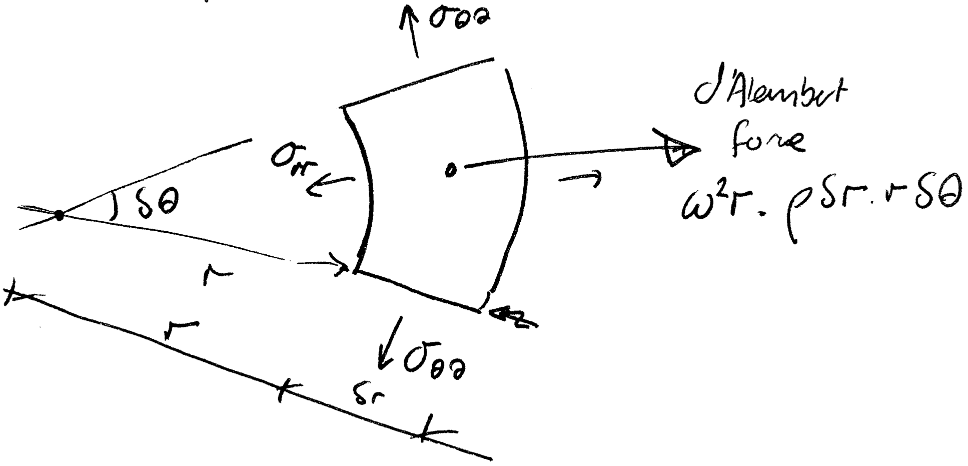
(c) To add tension, we want to uniformly increase σ_{xx} without changing anything else. If we have tension/unit depth T, we require $\frac{T}{d}$ added to σ_{xx}

$$\Rightarrow \phi_{new} = \phi + \frac{Ty^2}{2d}$$

This will not change σ_{xy} or σ_{yy}

2(a)

Radial equilibrium of small element



$$\rho\omega^2 r^2 \delta r \delta\theta + \frac{d(\sigma_{rr} . r\delta\theta)}{dr} . \delta r = 2\sigma_{\theta\theta} . \delta r . \frac{\delta\theta}{2}$$

$$\therefore \sigma_{\theta\theta} = \frac{d(r\sigma_{rr})}{dr} + \rho\omega^2 r^2$$

b(i) Design 1. No central hole $\Rightarrow B = 0$ (finite stresses, zero displacement etc.)

$$\therefore \sigma_{rr} = A - \frac{(3+\nu)}{8}\rho\omega^2 r^2$$

To satisfy equilibrium at outer boundary ($r = 0.1\text{m}$), $\sigma_{rr} = 0$

$$\therefore A = \frac{3.3}{8} \times 10^4 \times \omega^2 \times 0.1^2 = 41.25\omega^2 \quad \text{(units kg, m, s)}$$

$$\therefore \sigma_{rr} = 4125\,\omega^2 (0.1^2 - r^2)$$

$$\sigma_{\theta\theta} = 4125\,\omega^2 (0.1^2 - 0.576 r^2)$$

assume plane stress, $\sigma_{33} = 0$

Peak shear stress $= \frac{(\sigma_{rr} - \sigma_{33})}{2}$ at $r = 0$

Most candidates assumed, incorrectly! peak shear stress $= \frac{(\sigma_{rr} - \sigma_{\theta\theta})}{2}$

2(b)(i) cont.

Thus yield will occur at $\sigma_{rr}\big|_{r=0} = Y$

$$4125 \times 0.1^2 \omega^2 = 12 \times 10^6$$
$$\omega = 539 \text{ rad s}^{-1}$$

Design 2. $\sigma_{rr} = 0$ at $r = 0.01\,m$ and $0.1\,m$

At $r = 0.01$

$$0 = A - 10000B - 0.4125\omega^2 \qquad (1)$$

At $r = 0.1$

$$0 = A - 100B - 41.25\omega^2 \qquad (2)$$

Thus $B = 4.125 \times 10^{-3}\omega^2$; $A = 41.25\,\omega^2 + 0.4125\omega^2 = 41.66\,\omega^2$

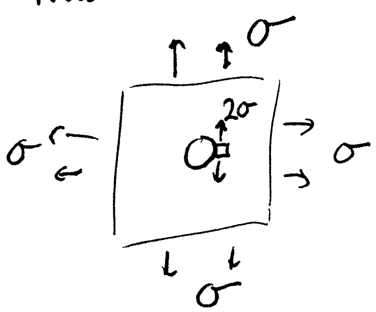
$$\sigma_{rr} = \omega^2\left[41.66 - \frac{4.125 \times 10^{-3}}{r^2} - 4125 r^2\right]$$

$$\sigma_{\theta\theta} = \omega^2\left[41.66 + \frac{4.125 \times 10^{-3}}{r^2} - 0.576 \times 4125 r^2\right]$$

Peak shear stress $= \dfrac{(\sigma_{\theta\theta} - \sigma_{33})}{2}$ at $r = 0.01$

$$\sigma_{\theta\theta} = \omega^2 \times 82.67 = 12 \times 10^6$$
$$\therefore \omega = 381 \text{ rad s}^{-1}$$

2(b)(ii) For a particular rotational speed, $\sigma_{\theta\theta}$ in case 2 is approx. double the value of $\sigma_{rr} = \sigma_{\theta\theta}$ in case 1. A hole in a biaxially loaded plate gives a stress concentration factor of 2, so this should be expected



Few candidates actually considered stresses for a given rotational speed, and did not make this connection

3 (a)

$$\phi = Cr\theta\sin\theta$$

$$\frac{\partial\phi}{\partial r} = C\theta\sin\theta$$

$$\frac{\partial^2\phi}{\partial r^2} = 0$$

$$\frac{\partial\phi}{\partial\theta} = Cr\sin\theta + Cr\theta\cos\theta$$

$$\frac{\partial^2\phi}{\partial\theta^2} = 2Cr\cos\theta - Cr\theta\sin\theta$$

$$\frac{\partial^2\phi}{\partial r\partial\theta} = C\sin\theta + C\theta\cos\theta$$

This gives stress components

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{2C}{r}\cos\theta$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = 0$$

$$\sigma_{r\theta} = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial\theta\partial r} = 0$$

(b) Need to check strains satisfy compatability, i.e. $\nabla^4\phi = 0$

i.e. $$\left\{\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\theta^2}\right\}\left\{\frac{2C}{r}\cos\theta\right\} = 0$$

$$= \frac{4C}{r^3}\cos\theta - \frac{2C}{r^3}\cos\theta - \frac{2C}{r^3}\cos\theta = 0 \quad \checkmark$$

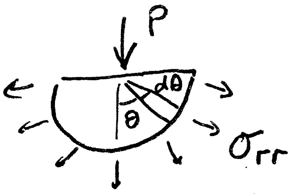
Candidates rarely checked this

Few candidates checked boundary conditions

3(b) cont. Also need to check B.C.'s At $\theta = \pm \frac{\pi}{2}$,

$$\sigma_{\theta\theta} = 0 \ , \ \sigma_{r\theta} = 0 \quad \checkmark$$

Now consider equilibrium with applied load on a semi-circle below load



Horizontal equilibrium by symmetry

For vertical equilibrium, $P + \int_{-\pi/2}^{\pi/2} \sigma_{rr} \cos\theta . r\, d\theta = 0$

$$P + 2C \int_{-\pi/2}^{\pi/2} \cos^2\theta = 0$$

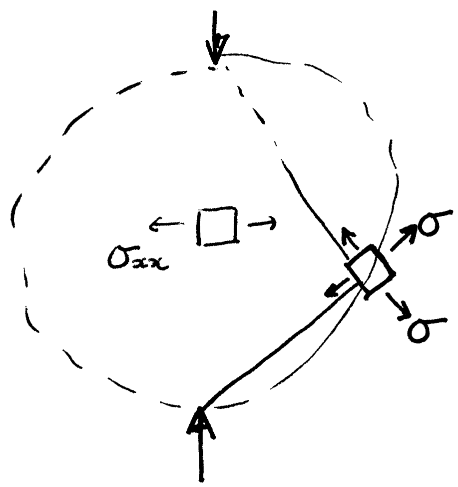
$$\therefore C = -\frac{P}{\pi}$$

(c)

Around the circle, $r = D\cos\theta$

$$\therefore \sigma_{rr} = \frac{-2P}{\pi D} \ ; \ \sigma_{\theta\theta} = 0$$

(d) Consider superposition of above solutions about two point loads at opposite ends of a diameter.



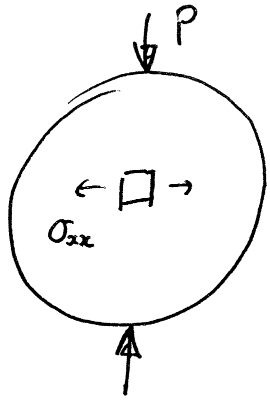
Uniform biaxial stress $\sigma = -\frac{2P}{\pi D}$

with $\sigma_{xx} = 0$

($\sigma_{\theta\theta} = 0$ for each load case)

3(d) cont.

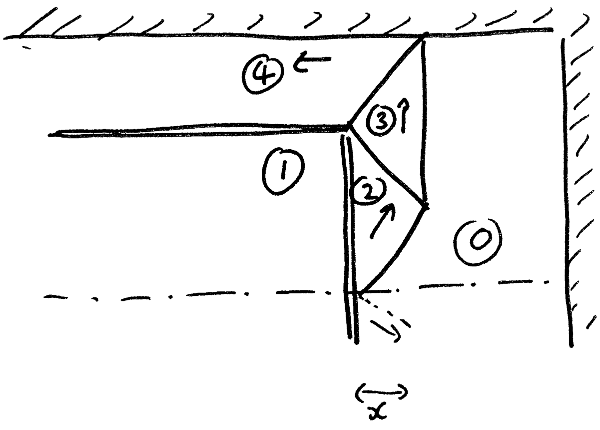
For the split cylinder, we need to superpose a uniform tension $\frac{2P}{\pi D}$ around circumference of circle – and hence everywhere in cylinder



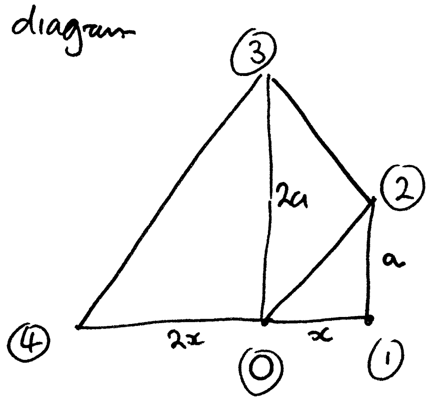
giving

$$\sigma_{xx} = \frac{2P}{\pi D}$$

4(a) Number regions



Velocity diagram



All $\times \left(\frac{v}{x}\right)$, so ① has velocity $\underrightarrow{v}$

Take unit depth, half problem, balance work done (WD) and energy dissipated (ED) to give upper bound

$$WD = v \cdot 2ap$$

$$ED = k\left[l_{02}v_{02} + l_{03}v_{03} + l_{23}v_{23} + l_{34}v_{34}\right]$$

where l_{ab} – length of ab boundary

v_{ab} – velocity discontinuity at ab boundary

$$\therefore ED = \frac{kv}{x} \cdot \left[l_{02}^2 + l_{03}^2 + l_{23}^2 + l_{34} \times (2l_{34})\right]$$

$$= \frac{kv}{x}\left[(x^2+a^2) + 4a^2 + (x^2+a^2) + (2x^2+2a^2)\right]$$

4(a) cont.

$$WD = ED$$

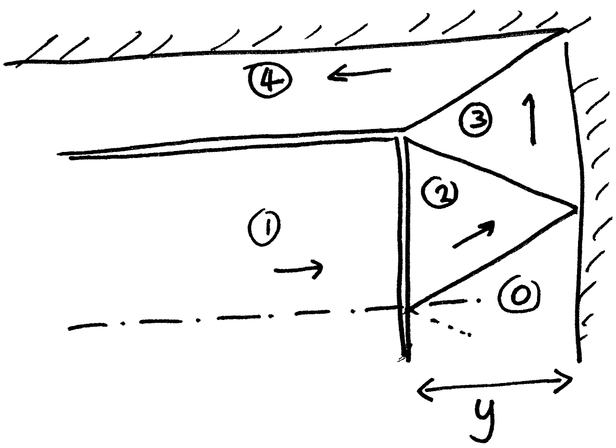
$$2ap v = \frac{kv}{x}\left(4x^2 + 8a^2\right)$$

$$\therefore p = 2k\left(\frac{x}{a} + \frac{2a}{x}\right)$$

This is an upper bound. Best estimate will be a minimum of p

(b) When plastic zone extends to the rear face, the boundary 03 will be along a lubricated boundary, and no energy will be dissipated here.

Using the same dislocation field, with y equal to distance from punch to end of die



4(b) cont.

$$\text{W.D.} = v.2ap$$

$$\text{E.D.} = \frac{kv}{y}\left[(x^2+a^2) + 0 + (x^2+a^2) + (2x^2+2a^2)\right]$$

As before, with l_{03} term set to zero

$$\therefore\ p = 2k\left(\frac{y}{a} + \frac{a}{y}\right)$$

(c) In (a), find minimum of $p = 2k(z + 2z^{-1})$ where $z = \frac{x}{a}$

$$\frac{dp}{dz} = 2k(1 - 2z^{-2}) = 0 \quad \text{for minimum}$$

$$\Rightarrow z = \sqrt{2}$$

$$x = \sqrt{2}\,a$$

giving $p = 4\sqrt{2}\,k$ – constant, valid for $y > \sqrt{2}\,a$

This step was omitted by many candidates, leading to confusion

Upper bound (b) will become preferable once

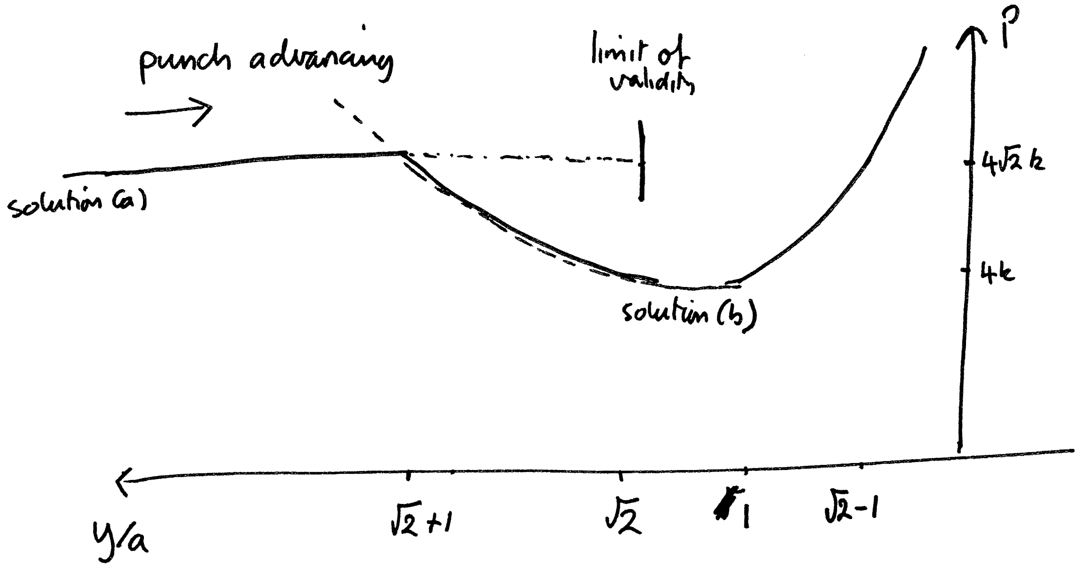
$$2k\left(\frac{y}{a} + \frac{a}{y}\right) = 4\sqrt{2}\,k$$

$$\left(\frac{y}{a}\right)^2 + 1 = 2\sqrt{2}\left(\frac{y}{a}\right)$$

$$\therefore \left(\frac{y}{a}\right) = \frac{2\sqrt{2} \pm \sqrt{8-4}}{2} = \sqrt{2} \pm 1$$

i.e. first occurs when $z = \sqrt{2} + 1$

4(c) cont. Graph of solutions



3C7 2010: Answers

1.(a) $\sigma_{xx} = \dfrac{w}{10d^3}\left[6d^2y - 15l^2y + 60x^2y - 40y^3\right];$

$\sigma_{yy} = \dfrac{w}{2d^3}\left[-d^3 - 3yd^2 + 4y^3\right];$

$\sigma_{xy} = \dfrac{w}{2d^3}\left[-12xy^2 + 3xd^2\right]$

2.(b.i) $539\,\mathrm{rad\,s^{-1}}$; $381\,\mathrm{rad\,s^{-1}}$

3.(a) $\sigma_{rr} = \dfrac{2C}{r}\cos\theta$; $\sigma_{\theta\theta} = 0$; $\sigma_{r\theta} = 0$

(d) $\sigma_{xx} = \dfrac{2P}{\pi D}$

4.(b) One mechanism gives $p = 2k\left(\dfrac{y}{a} + \dfrac{a}{y}\right)$

(c) $z = \sqrt{2} + 1$