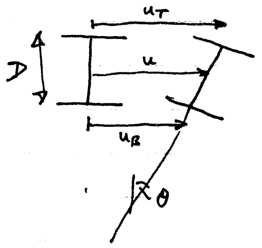
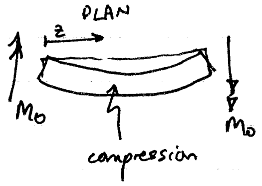
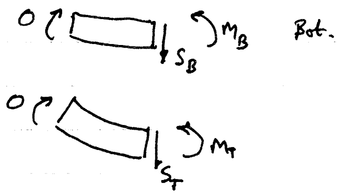
3D4

3D4.

Q1.





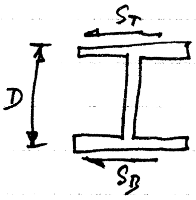


$$S = \frac{dM}{dz} \quad \text{and} \quad M_B = -EI_f \frac{d^2u_B}{dz^2} \qquad M_T = -EI_f \frac{d^2u_T}{dz^2}$$

$$u_B = u - \frac{D}{2}\theta \qquad u_T = u + D/_2\,\theta$$

$$S_B = \frac{dM_B}{dz} = -EI_f \frac{d^3u_B}{dz^3} = -EI_f\left[\frac{d^3u}{dz^3} - \frac{D}{2}\frac{d^3\theta}{dz^3}\right]$$

$$S_T = \frac{dM_T}{dz} = -EI_f \frac{d^3u_T}{dz^3} = -EI_f\left[\frac{d^3u}{dz^3} + \frac{D}{2}\frac{d^3\theta}{dz^3}\right]$$



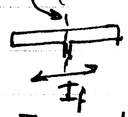
Extra twisting moment:

$$S_T\frac{D}{2} - S_B\frac{D}{2} = \frac{D}{2}EI_f\left[-u''' - \frac{D}{2}\theta''' + u''' - \frac{D}{2}\theta'''\right] = -\frac{D^2}{2}EI_f\theta'''$$

$$= -E\Gamma\theta'''$$

↑ restrained warping torsion const

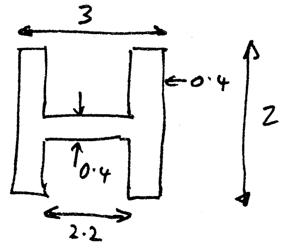
$\therefore \quad \Gamma = \dfrac{I_f D^2}{2}$ where I_f = 2nd mom. of area of flange about its own major axis (= the Ibeam's minor axis)



D = distance between centres of flanges.

1.

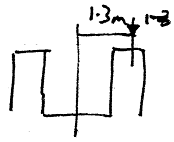
(b) i) $J = \Sigma \frac{bt^3}{3}$



$= \frac{(2{\cdot}2+2+2)(0{\cdot}4)^3}{3} = \frac{(6{\cdot}2)(0{\cdot}4)^3}{3} = \underline{\underline{0{\cdot}1323 \text{ m}^4}}$

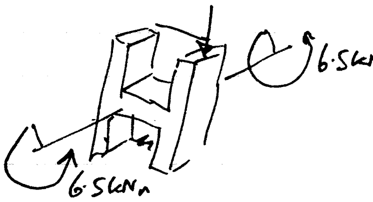
$T = GJ\phi \qquad \phi = \frac{\theta}{L}$

T is constat



applied couple = $10\text{kN} \times (1{\cdot}3\text{m})$
$= 13 \text{kNm}$

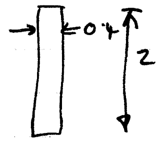
resisted by $T = \frac{13}{2} = 6{\cdot}5 \text{kNm}$ each way



$G = \frac{E}{2(1+\nu)} = \frac{30 \times 10^9 \text{ Nm}^{-2}}{2(1{\cdot}2)}$
$= \underline{\underline{12{\cdot}5 \times 10^9 \text{ Nm}^{-2}}}$

$\phi = \frac{\theta}{L} = \frac{T}{GJ} \quad \rightarrow \quad \theta = \frac{TL}{GJ} = \frac{(6500)\text{Nm}(6\text{m})}{(12{\cdot}5\times10^9)\text{Nm}^{-2}(0{\cdot}1323\text{m}^4)}$
$= \underline{\underline{2{\cdot}36 \times 10^{-5} \text{ radians}}}.$

ii) $\Gamma = \frac{I_f d^2}{2}$ $\qquad I_f = \frac{bd^3}{12}$
$= \frac{(0{\cdot}4)(2)^3}{12}$
$= \underline{\underline{0{\cdot}2667 \text{ m}^4}}$



$\therefore \quad \Gamma = \frac{I_f d^2}{2}$ $\qquad d$ = distance between centres of flanges
$= 2{\cdot}6\text{m}$

$= \frac{(0{\cdot}2667)(2{\cdot}6)^2}{2} = \underline{\underline{0{\cdot}901 \text{ m}^6}}$

$T = GJ\phi - E\Gamma \frac{d^2\phi}{dz^2}$

(b ii) cont'd.

At ~~root~~ end, no twist, so $\theta = 0$ at $z=0$

At centre, $\frac{du}{dz} = 0$ so $\theta' = 0$ at $z = L/2$ (restrained warping)

At end, no moment in flange, so $\theta'' = 0$ at $z = 0$ (free to warp)

$$\therefore \quad \phi = \frac{T}{GJ} + Ae^{-\alpha z} + Be^{\alpha z} \qquad \alpha = \sqrt{\frac{GJ}{E\Gamma}}$$

$$\theta = \frac{T}{GJ}(z+c) + A_1 e^{-\alpha z} + B_1 e^{\alpha z}$$

const of integration.

$\theta'' = 0$ at $z = 0$

$$\rightarrow \quad \alpha^2(A_1 + B_1) = 0 \quad \rightarrow \quad B_1 = -A_1$$

$\theta' = 0$ at $z = L/2$

$$\rightarrow \quad \frac{T}{GJ} - \alpha A_1 e^{-\alpha L/2} + \alpha B_1 e^{\alpha L/2} = 0$$

solve for A_1, B_1.

$$\frac{T}{GJ} - \alpha A_1 e^{-\alpha L/2} - \alpha A_1 e^{\alpha L/2} = 0$$

$$\frac{T}{GJ} - \alpha A_1 \left(e^{-\alpha L/2} + e^{\alpha L/2}\right) = 0$$

$$\Rightarrow \underline{\underline{A_1 = \frac{T}{\alpha GJ} \frac{1}{(e^{-\alpha L/2} + e^{\alpha L/2})}}} \qquad \text{and } B_1 = -A_1$$

Also, need c.

$\theta = 0$ at $z = 0$

$$0 = \frac{T}{GJ}c + A_1 + B_1 \quad \rightarrow \quad c = 0 \quad (\text{since } A_1 + B_1 = 0)$$

$$\therefore \quad \theta = \frac{T}{GJ}(z) + \frac{T}{GJ}\left[\frac{1}{\alpha(e^{-\alpha L/2} + e^{\alpha L/2})}(e^{-\alpha z} - e^{\alpha z})\right]$$

$\therefore$ When $z = L/2$

$$\theta = \frac{T}{GJ}(L/2) + \frac{T}{GJ}\left[\frac{1}{\alpha}\frac{(e^{-\alpha L/2} - e^{\alpha L/2})}{(e^{-\alpha L/2} + e^{\alpha L/2})}\right]$$

$$= \theta_{sv}\left[1 - \frac{1}{\alpha L/2}\frac{(e^{\alpha L/2} - e^{-\alpha L/2})}{(e^{-\alpha L/2} + e^{\alpha L/2})}\right]$$

$$= \theta_{sv}\left[1 - \frac{1}{(\alpha L/2)}\tanh(\alpha L/2)\right]$$

1b)
cont'd.

$$\frac{\alpha L}{2} = \frac{L}{2}\sqrt{\frac{GJ}{E\Gamma}} = 6m\left(\frac{1}{2(1+0.2)} \frac{0.1323}{0.901}\right)^{1/2}$$

$$= 1.4841$$

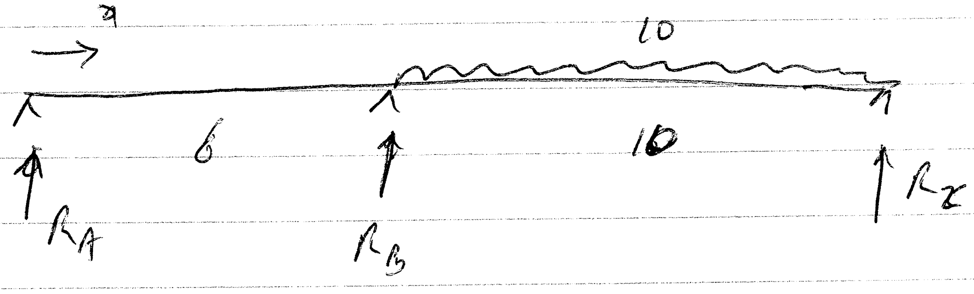
$$\frac{1}{(\alpha L/2)} \tanh(\alpha L/2) = 0.608$$

$$\therefore \quad \theta = \theta_{sv}\left(1 - 0.608\right)$$

$$= 2.36 \times 10^{-5}\,(1 - 0.608) = \underline{\underline{0.925 \times 10^{-5} \text{ radians}}}$$

3D4

Q2.



$$-EI\frac{d^2v}{dx^2} = R_A.x + R_B\{x-6\} - \overset{5}{\cancel{10}}\frac{\{x-6\}^2}{\cancel{2}}$$

$$-EIv = R_A\frac{x^3}{6} + R_B\frac{\{x-6\}^3}{6} - \frac{5}{12}\{x-6\}^4 + Ax + B$$

$x=0, \quad v=0 \quad \Rightarrow \quad B=0$

$x=6, \quad v=0$

$$\Rightarrow \quad R_A.36 + A.6 = 0$$

$$\Rightarrow \quad A = -6R_A$$

$x=16 \quad v=0$

$$0 = R_A.\frac{4096}{6} + R_B\frac{1000}{6} - \frac{50000}{12} \cancel{4} - 96R_A$$

$$\Rightarrow \quad 3520\,R_A + 1000\,R_B = 25000 \qquad (1)$$

Take moments about C

$$16\,R_A + 10\,R_B = 10 \times 10 \times 5 = 500 \qquad (2)$$

$$\Rightarrow \quad R_B = 50\cancel{0} - 1.6\,R_A$$

3D4
Q2

$$3520\,R_A + 5.10^4 - 1600\,R_A = 25000$$

$$1920\,R_A = \cancel{475000}\ -25000$$

$$\cancel{R_A = 247.4\ \text{KN}}$$

$$R_A = -13.02\ \text{KN}$$ (agree with program
$$R_B = 70.83\ \text{KN}$$

$$R_C = 100 - 70.83 + 13.02 = 42.19\ \text{KN}$$

(b)

$$-EIv = \frac{-13.02}{6}x^3 + \frac{70.83\{x-6\}^3}{6} - \frac{5}{12}\{x-6\}^4$$
$$+13.02\,x.6$$

$$v = \frac{1}{EI}\left(2.17x^3 - 11.80\{x-6\}^3 + 0.417\{x-6\}^4 + \overset{78.12x}{\cancel{13.02x}}\right)$$

c) $$\frac{dv}{dx} = \frac{1}{EI}\left(6.51x^2 - 35.42\{x-6\}^2 + 1.67\{x-6\}^3 - \overset{78.12}{\cancel{13.02}}\right)$$

$= 0$ at maximum

Expect maximum for $x > 6$ $\therefore$ expand Macaulay brackets

$$0 = 1.67x^3 - 30x^2 + 180x - 360$$ (from $\{x-6\}^3$ term
$$- 35.42x^2 + 425x - 1275$$ (from $\{x-6\}^2$ term
$$+ 6.51x^2 \qquad - 78.12$$
$$1.67x^3 - 58.91x^2 + 605x - 1713 = 0$$

2 c) cont'd.

$$x \approx 11.4\,m \Rightarrow \frac{dv}{dx} \approx 0.$$

$$v = \frac{1}{EI}\left(2.17x^3 - 11.8\{x-6\}^3 + 0.417\{x-6\}^4 + 78.12x\right)$$

10^5 kNm²

$$\Rightarrow \underline{\underline{v = 0.026\,m}}$$

3(a) Marks will be obtained for explaining how the total potential energy function can be expanded around any equilibrium position such that the Taylor Series expansion there is

$$\Pi = \underbrace{\text{Const}}_{\substack{\text{arbitrary}\\ \text{datum}}} + \underbrace{\text{linear}}_{\substack{\text{zero}\\ \text{at equilib}}} + \underbrace{\text{quadratic}}_{\substack{\text{first significant}\\ \text{term}}} + \text{h.o.t}$$

thus, for small perturbations, the energy is a quadratic form expressible as $\frac{1}{2}\, \delta x^T \frac{\partial^2 \Pi}{\partial x_i \partial x_j} \delta x = \frac{1}{2} \delta x^T K \delta x$

where K is the total tangent stiffness matrix
(the matrix of curvatures of the total potential energy function)

The principal directions of K are its eigenvectors, and the associated eigenvalues are the stiffnesses in these directions (where "direction" means "deflected shape").
This is true under <u>any</u> loading.
At certain critical loads, an eigenvalue may become zero. This means there is no stiffness in the associated eigenvector deformed shape, such that it can move in that direction (shape) without resistance (i.e. it buckles into that buckling mode).

Such static eigenvectors are eigenvectors of K alone, whereas dynamic eigenvectors depend on the mass matrix also (from generalised e.v. problem $[-\omega^2 M + K]x = 0$)

thus dynamic evecs usually different to static evecs, (unless $M \propto K$ or $\omega = 0$)

3 b) Rayleigh Quotient assumptions

— trivial solution $w=0$ is equilib

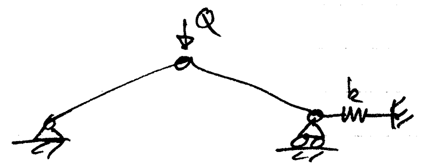
- small deflection theory is used

- axial deflections y depend on square of lateral w

- only a 1-D space is explored $w = a\,\psi(x)$

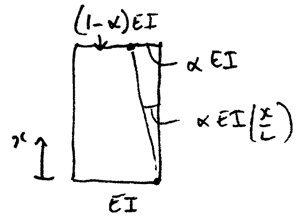
- clear separation into "axial" and "lateral"

Doesn't work for snap-through bifurcations, for example:



3 c i) Rayleigh Quotient $\quad P_{RR} = \dfrac{\int_0^L \text{"}EI\text{"} (\psi'')^2 \, dx}{\int_0^L (\psi')^2 \, dx}$

$\text{"}EI\text{"} = EI\left(1 - \alpha \dfrac{x}{L}\right)$



Let $w = ax^2$

$\psi = x^2$

$\psi' = 2x$

$\psi'' = 2$

$$2U(\psi) = \int_0^L \text{"}EI\text{"} (\psi'')^2 dx = \int_0^L EI\left(1 - \alpha\left(\frac{x}{L}\right)\right)[4]\, dx$$

$$= 4EI\left[L - \frac{\alpha}{L}\frac{L^2}{2}\right] = \underline{\underline{4EIL\left[1 - \frac{\alpha}{2}\right]}}$$

$$2y(\psi) = \int_0^L (\psi')^2 dx = \int_0^L 4x^2 dx = \underline{\underline{\frac{4L^3}{3}}}$$

$$P_{RR} = \frac{U(\psi)}{y(\psi)} = \frac{4EIL\left[1 - \alpha/2\right]}{4L^3/3} = \underline{\underline{3\frac{EI}{L^2}\left[1 - \frac{\alpha}{2}\right]}}$$

[Check, $\alpha = 0 \rightarrow 3EI/L^2$ cf Euler $\dfrac{\pi^2 EI}{(2L)^2} = \dfrac{\pi^2}{4}\dfrac{EI}{L^2} \sim 2.5\dfrac{EI}{L^2}$
(uniform)

$\therefore P_{RR} > P_{Euler}$ ✓
Upper Bound.]

3 (c) (ii) Let $w = a\left(x + \frac{b}{L}x^2\right) = a\psi$

with $\psi = x + \frac{b}{L}x^2$

$\psi' = 1 + 2\frac{b}{L}x$

$\psi'' = \frac{2b}{L}$

and $\theta_0 = \psi'(x=0) = 1$

Strain Energy $U(\psi) = \int_0^L \frac{1}{2} EI (\psi'')^2 dx + \frac{1}{2} G (\psi'|_0)^2$

$= \frac{1}{2}\int_0^L EI\left(\frac{4b^2}{L^2}\right) dx + \frac{1}{2} G$

$= \frac{1}{2}\left[4\frac{EI}{L}b^2 + G\right] = \frac{1}{2}\left[\frac{4EI b^2}{L} + \frac{3EI}{L}\right]$

$$= \frac{1}{2}\left(\frac{EI}{L}\right)\left[4b^2 + 3\right]$$

Dist. moved by P $y(\psi) = \frac{1}{2}\int_0^L (\psi')^2 dx$

$= \frac{1}{2}\int_0^L \left(1 + \frac{2b}{L}x\right)^2 dx = \frac{1}{2}\int_0^L \left(1 + \frac{4b}{L}x + \frac{4b^2}{L^2}x^2\right) dx$

$= \frac{1}{2}\left[L + \frac{2bL^2}{L} + \frac{4}{3}b^2\frac{L^3}{L^2}\right] = \frac{1}{2}L\left[1 + 2b + \frac{4}{3}b^2\right]$

so $$P_{RR} = \frac{U(\psi)}{y(\psi)} = \frac{EI}{L^2}\left[\frac{4b^2+3}{1+2b+\frac{4}{3}b^2}\right]$$

Minimise wrt b

$\frac{dP_{RR}}{db} = 0 \Rightarrow \frac{8b\left[1+2b+\frac{4}{3}b^2\right] - \left[4b^2+3\right]\left[2+\frac{8}{3}b\right]}{[\quad]^2} = 0$

$\Rightarrow \cancel{8b} + 16b^2 + \cancel{\frac{32b^3}{3}} - 8b^2 - \cancel{\frac{32b^3}{3}} - 6 - \cancel{\frac{8b(3)}{3}}$

$8b^2 - 6 = 0 \rightarrow 4b^2 = 3 \rightarrow \underline{\underline{b = \pm\sqrt{3}/2}}$

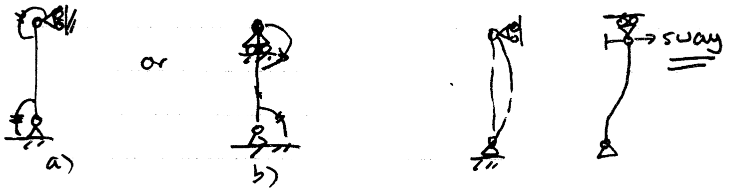
By inspection, +ve will be worst ✓, but check.

$+ \rightarrow \frac{4b^2+3}{1+2b+\frac{4}{3}b^2} = \frac{6}{1+\sqrt{3}+1} = \frac{6}{2+\sqrt{3}} = 1.6077. \rightarrow \boxed{P_{RR} = 1.61\frac{EI}{L^2}}$

$\left(-ve \Rightarrow \frac{4b^2+3}{1+2b+\frac{4}{3}b^2} = \frac{6}{1-\sqrt{3}+1} = \frac{6}{2-\sqrt{3}} = 22.4\right)$

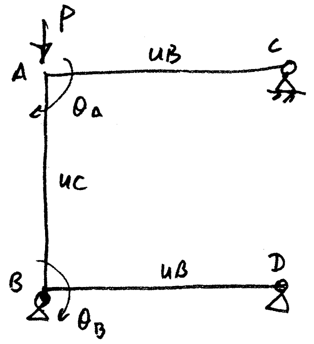
4(a). Effective length l_e of a column is defined via its elastic critical load for buckling $P_{cr} = \frac{\pi^2 EI}{l_e^2}$ thus $l_e = \pi \sqrt{\frac{EI}{P_{cr}}}$.

Distant parts of buildings affect the effective length since need to know if top is braced or unbraced against sway.



Construction details that might provide sway bracing include shear walls, cross-braced bays or stiff shear walls around say lift shafts + service cores.

4(b)



EI_{major} for UB ; $610 \times 305 \times 149$ = 125 900 ~~$305 \times 127 \times 48$~~ = ~~9575~~ cm4

UC : $305 \times 305 \times 118$ = 27670 cm4

Stiffness matrix $K = \begin{bmatrix} s k_{uc} + 3k_{uB} & sc\, k_{uc} \\ sc\, k_{uc} & s k_{uc} + 3 k_{uB} \end{bmatrix}$ for $\begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix}$

for end assumed pinned.

$k = \frac{EI}{L}$ UC $\propto \frac{27670}{3} = 9223$

UB $\propto \frac{125\,900}{7} = 17{,}990$

$\therefore \frac{3k_{uB}}{k_{uc}} = \frac{3(17{,}990)}{9223} = 5.85$

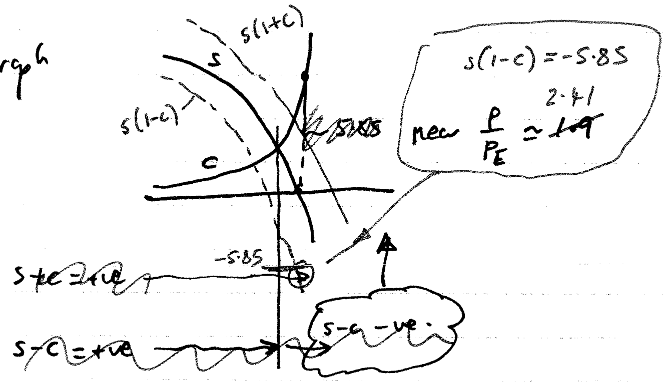
$\det K = 0 \Rightarrow \therefore (k_{uc})^2\left[(s+5.85)^2 - s^2c^2\right] = 0$

$\therefore$ Require $s + 5.85 = \pm sc$

$\therefore$ $s + sc = -5.85$ $s(1+c) = -5.85$

or $s - sc = -5.85$ $s(1-c) = -5.85$

graph



$\therefore$ $P \approx \frac{\pi^2 EI}{L^2}$ (~~1.9~~ 2.41)

$= \frac{\pi^2 EI}{(L/\sqrt{2.41})^2}$

$\therefore$ $l_e = \frac{L}{\sqrt{2.41}}$ = ~~2.18 m~~ $\frac{L}{1.55}$

$= 0.644L = 1.935$

so $P_{cr} = \frac{\pi^2 EI}{1.935^2} = \frac{\pi^2 (205 \times 10^9\ N/m^2)(27670 \times 10^{-8}\ m^4)}{(2.18)^2\ m^2}$ ~~1.18~~

$=$ ~~1.18~~ 1.495 $\times 10^8$ N = ~~118 x 10^3 kN~~

149.5 x 10^3 kN

4b ii)

If loading of column is from loading of AC

- first need to calculate load carried by AB
  - and as interested in when AB buckles, can assume end A is pinned (as column AC will provide no rotational restraint to end A of beam AC)

- but, more specifically, any end moment at A in AC (prior to Euler buckling of AB) could precipitate lateral torsional buckling of AB.
  (and should really consider interaction of LTB with minor axis (out of plane) buckling).

4b iii) Again - loads on BD could cause LTB of column AB by creating major axis moment at B.

4 b ~~ii~~ ii) and iii) Also, if loads on beams AC or BD are sufficient to cause plastic hinges to form at A and/or B, can no longer assume that these beams provide rotational stiffness there, hence effective length of AB $\rightarrow$ 3m (full length).