3D7: Finite Element Method
— Crib for 2010—

1. (a) Multiply differential equation with weight function v and integrate

$$\int_0^L v\alpha \frac{du}{dx}\, dx - \int_0^L v\frac{d}{dx}\left(\beta\frac{du}{dx}\right)\, dx = \int_0^L vf\, dx$$

Integrate second term on the left hand side by parts

$$\int_0^L v\alpha \frac{du}{dx}\, dx + \int_0^L \frac{dv}{dx}\beta\frac{du}{dx}\, dx - v\beta\frac{du}{dx}\Big|_0^L = \int_0^L vf\, dx$$

Insert Neumann boundary conditions and set $v(0) = 0$

$$\int_0^L v\alpha \frac{du}{dx}\, dx + \int_0^L \frac{dv}{dx}\beta\frac{du}{dx}\, dx - v(L)h = \int_0^L vf\, dx$$

Restrictions on v:

- $v = 0$ at $x = 0$
- v must be sufficiently smooth

(b)

$$\boldsymbol{k}_e = \boldsymbol{k}_1 + \boldsymbol{k}_2$$

$$\boldsymbol{k}_1 = \int_0^l \alpha \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \end{bmatrix} dx = \alpha \int_0^l \begin{bmatrix} -\frac{x}{l} + 1 \\ \frac{x}{l} \end{bmatrix} \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix} dx$$

$$= \alpha \int_0^l \begin{bmatrix} \frac{x}{l^2} - \frac{1}{l} & -\frac{x}{l^2} + \frac{1}{l} \\ -\frac{x}{l^2} & \frac{x}{l^2} \end{bmatrix} dx = \alpha \begin{bmatrix} \frac{x^2}{2l^2} - \frac{x}{l} & -\frac{x^2}{2l^2} + \frac{x}{l} \\ -\frac{x^2}{2l^2} & \frac{x^2}{2l^2} \end{bmatrix}\Bigg|_0^l$$

$$= \alpha \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\boldsymbol{k}_2 = \int_0^l \beta \boldsymbol{B}^T \boldsymbol{B}\, dx = \beta \int_0^l \begin{bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{bmatrix} \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} dx$$

$$= \beta \int_0^l \begin{bmatrix} \frac{1}{l^2} & -\frac{1}{l^2} \\ -\frac{1}{l^2} & \frac{1}{l^2} \end{bmatrix} dx = \frac{\beta}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow \boldsymbol{k}_e = \boldsymbol{k}_1 + \boldsymbol{k}_2 = \begin{bmatrix} -\frac{\alpha}{2} + \frac{\beta}{l} & \frac{\alpha}{2} - \frac{\beta}{l} \\ -\frac{\alpha}{2} - \frac{\beta}{l} & \frac{\alpha}{2} + \frac{\beta}{l} \end{bmatrix}$$

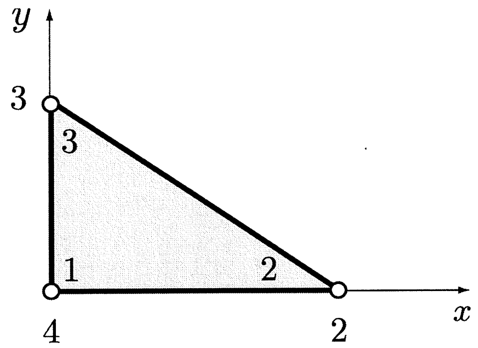
(c) Apply integration by parts to the first term

$$-\int_0^L \alpha \frac{dv}{dx} u\, dx + \alpha v u \Big|_0^L + \int_0^L \frac{dv}{dx} \beta \frac{du}{dx}\, dx - v(L)h = \int_0^L v f\, dx$$

Considering that $v = 0$ at $x = 0$

$$-\int_0^L \alpha \frac{dv}{dx} u\, dx + \alpha v(L) u(L) + \int_0^L \frac{dv}{dx} \beta \frac{du}{dx}\, dx - v(L)h = \int_0^L v f\, dx$$

2. It is sufficient to consider only one of the elements in the triangular domain. Consider element 1 (top element) in a local coordinate system, which gives simple expressions for the shape functions



$$N_1 = 1 - \frac{x}{3} - \frac{y}{2} \qquad N_2 = \frac{x}{3} \qquad N_3 = \frac{y}{2}$$

(a)

$$\boldsymbol{f}^1 = \int_0^2 -10 \cdot \begin{bmatrix} \frac{y}{2} \\ 0 \\ \left(1 - \frac{y}{2}\right) \end{bmatrix} dy = -10 \cdot \begin{bmatrix} \frac{y^2}{4} \\ 0 \\ \left(y - \frac{y^2}{4}\right) \end{bmatrix} \Bigg|_0^2 = \begin{bmatrix} -10 \\ 0 \\ -10 \end{bmatrix}$$

The element source vector of element 2 can be deduced from the element source vector of element 1. No need for computation. The element source vectors assembled yield the global source vector

$$f^T = \begin{bmatrix} -10 & 0 & -10 & -20 \end{bmatrix}$$

(b) Conductance matrix for element 1

$$\boldsymbol{k}^1 = A^1 \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B}$$

where $A^1 = 3$ is the area of the element. Furthermore,

$$\boldsymbol{B} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

and

$$\boldsymbol{D} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

so that

$$\boldsymbol{k}^1 = A^1 \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} = \begin{bmatrix} -\frac{13}{4} & -1 & \frac{9}{4} \\ -1 & 1 & 0 \\ -\frac{9}{4} & 0 & \frac{9}{4} \end{bmatrix}$$

The global conductance matrix for element 2 can be deduced from the conductance matrix of element 1. The element conductance matrices assembled yield the global conductance matrix

$$\boldsymbol{K} = \begin{bmatrix} — & — & — & — \\ — & — & — & — \\ — & — & \frac{9}{4} & -\frac{9}{4} \\ — & — & -\frac{9}{4} & \left(\frac{13}{4} + \frac{13}{4} \cdot 2\right) \end{bmatrix}$$

The components that correspond to nodes with prescribed temperature have been not assembled (i.e., —). The (4, 4) component has contributions from elements 1 and 2. The contribution of element 2 is

$$\underbrace{\frac{k_2}{k_1}}_{2} \frac{13}{4}$$

(c)

$$\begin{bmatrix} \frac{9}{4} & -\frac{9}{4} \\ \frac{-9}{4} & \frac{39}{4} \end{bmatrix} \begin{bmatrix} T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} -10 \\ -20 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} -8.44 \\ -4 \end{bmatrix}$$

Flux across CB

$$\boldsymbol{q} = -k\boldsymbol{\nabla}T \quad \text{or} \quad \boldsymbol{q} = -\boldsymbol{DBT}$$

$$\boldsymbol{\nabla}T = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -4 \\ 10 \\ -8.44 \end{bmatrix} = \begin{bmatrix} 4.667 \\ 2.222 \end{bmatrix}$$

$$\Rightarrow \boldsymbol{q} = \begin{bmatrix} 14.0 \\ -6.667 \end{bmatrix} \qquad \Rightarrow q_n = \boldsymbol{q} \cdot \boldsymbol{n} = \begin{bmatrix} 14.0 \\ -6.667 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = 1.7285$$

In order to improve accuracy much finer mesh needed!

3. (a)

i. The shape functions of the element are

$$N_1 = (1 - \frac{x}{5})(1 - \frac{y}{5})/4$$
$$N_2 = (1 + \frac{x}{5})(1 - \frac{y}{5})/4$$
$$N_3 = (1 + \frac{x}{5})(1 + \frac{y}{5})/4$$
$$N_4 = (1 - \frac{x}{5})(1 + \frac{y}{5})/4$$

with the corresponding derivatives

$\frac{\partial N_1}{\partial x} = -\frac{1}{20}(1 - \frac{y}{5})$	$\frac{\partial N_1}{\partial y} = -\frac{1}{20}(1 - \frac{x}{5})$
$\frac{\partial N_2}{\partial x} = \frac{1}{20}(1 - \frac{y}{5})$	$\frac{\partial N_2}{\partial y} = -\frac{1}{20}(1 + \frac{x}{5})$
$\frac{\partial N_3}{\partial x} = \frac{1}{20}(1 + \frac{y}{5})$	$\frac{\partial N_3}{\partial y} = \frac{1}{20}(1 + \frac{x}{5})$
$\frac{\partial N_4}{\partial x} = -\frac{1}{20}(1 + \frac{y}{5})$	$\frac{\partial N_4}{\partial y} = \frac{1}{20}(1 - \frac{x}{5})$

The strain components are computed with

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial}{\partial x}(N_1 u_{1x} + N_2 u_{2x} + N_3 u_{3x} + N_4 u_{4x})$$
$$\epsilon_{xx}(y = 5) = \frac{1}{10}u_{3x} - \frac{1}{10}u_{4x} = \frac{1}{5}$$

$$2\epsilon_{xy} = \frac{\partial u_x}{\partial y} + \underbrace{\frac{\partial u_y}{\partial x}}_{0}$$

$$2\epsilon_{xy}(y = 5) = -\frac{1}{20}(1 - \frac{x}{5}) \cdot 1 - \frac{1}{20}(1 + \frac{x}{5}) \cdot (-1) + \frac{1}{20}(1 + \frac{x}{5}) \cdot 1 + \frac{1}{20}(1 - \frac{x}{5}) \cdot (-1)$$
$$2\epsilon_{xy}(y = 5) = \frac{x}{25}$$

ii. The nodal forces belong to a beam with constant bending moment. There should not be any shear deformations (ϵ_{xy}).

(b)

i. The shape function can be constructed with the tensor product method

$$N_1^{1d}(x) = (1 - \frac{x}{2})$$
$$N_1^{1d}(y) = \frac{y}{2}$$
$$N_1^{1d}(z) = \frac{z}{3}$$
$$\Rightarrow N_1(x,y,z) = N_1^{1d}(x) \times N_1^{1d}(y) \times N_1^{1d}(z) = (1 - \frac{x}{2})\frac{y}{2}\frac{z}{3}$$

ii. The stiffness matrix should be integrated with $2 \times 2 \times 2$ integration points. In the stiffness matrix of the undistorted element polynomials up to degree four are possible.

4. (a)

i. Consider the time stepping equation

$$y_{n+1} = y_n + (1-\theta)\Delta t \dot{y}_n + \theta \Delta t \dot{y}_{n+1}$$

and its time derivative

$$\dot{y}_{n+1} = \dot{y}_n + (1-\theta)\Delta t \ddot{y}_n + \theta \Delta t \ddot{y}_{n+1}$$

Both equations are combined to eliminate $\dot{y}_{n+1}$

$$y_{n+1} - y_n = (1-\theta)\Delta t \dot{y}_n + \theta \Delta t \dot{y}_n + (1-\theta)\theta \Delta t^2 \ddot{y}_n + \theta^2 \Delta t^2 \ddot{y}_{n+1}$$

so that

$$\theta^2 \Delta^2 \ddot{y}_{n+1} = y_{n+1} - y_n - \Delta t \dot{y}_n - (1-\theta)\theta \Delta t^2 \ddot{y}_n$$

Next, the preceding equations are inserted into the semi-discrete finite element equation

$$\theta^2 \Delta t^2 M \ddot{a}_{n+1} + \theta^2 \Delta t^2 K a_{n+1} = \theta^2 \Delta t^2 b_{n+1}$$
$$\Rightarrow \quad M\left(a_{n+1} - a_n - \Delta t \dot{a}_n - (1-\theta)\theta \Delta t^2 \ddot{a}_n\right) + \theta^2 \Delta t^2 K a_{n+1} = \theta^2 \Delta t^2 b_{n+1}$$

Solve the following equation to obtain a_{n+1}

$$(M + \theta^2 \Delta t^2 K) a_{n+1} = \theta^2 \Delta t^2 b_{n+1} + M\left(a_n + \Delta t \dot{a}_n + (1-\theta)\theta \Delta t^2 \ddot{a}_n\right)$$

ii. To avoid solving a system of equations, set $\theta = 0$ and lump the mass matrix.

(b)

i. Original element size h_1. Element size of the refined mesh h_2.
1D: Double number of elements $\Rightarrow h_2 = \frac{1}{2}h_1$. If the time step $\propto h^2$, this will lead to a factor of 4 reduction in critical time step.

2D: Double number of elements $\Rightarrow h_2 = \frac{1}{\sqrt{2}}h_1$. If the time step $\propto h^2$, this will lead to a factor of 2 reduction in critical time step.

3D: Double number of elements $\Rightarrow h_2 = \frac{1}{2^{1/3}}h_1$. If the time step $\propto h^2$, this will lead to a factor of 1.59 reduction in critical time step.

ii. Orthogonality property lost since matrix changes from time step to time step.

The list of numerical answers for 3D7

1b)

$$\boldsymbol{k}_e = \begin{bmatrix} -\frac{\alpha}{2} + \frac{\beta}{l} & \frac{\alpha}{2} - \frac{\beta}{l} \\ \\ -\frac{\alpha}{2} - \frac{\beta}{l} & \frac{\alpha}{2} + \frac{\beta}{l} \end{bmatrix}$$

2a)

$$\boldsymbol{f}^T = \begin{bmatrix} -10 & 0 & -10 & -20 \end{bmatrix}$$

2b)

$$\boldsymbol{K} = \begin{bmatrix} \text{---} & \text{---} & \text{---} & \text{---} \\ \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \\ \text{---} & \text{---} & \frac{9}{4} & -\frac{9}{4} \\ \\ \text{---} & \text{---} & -\frac{9}{4} & \left(\frac{13}{4} + \frac{13}{4} \cdot 2\right) \end{bmatrix}$$

2c)

$$\begin{bmatrix} T_3 \\ \\ T_4 \end{bmatrix} = \begin{bmatrix} -8.44 \\ \\ -4 \end{bmatrix}$$

$$q_n = 1.7285$$

3a)

$$\epsilon_{xx} = \frac{1}{5}$$

$$\epsilon_{xx} = \frac{1}{50}$$

4b) 1D: Reduction factor 4
2D: Reduction factor 2
3D: Reduction factor 1.59