2010 Solutions PART IIA 3F1 – SIGNALS AND SYSTEMS Dr J. Gonçalves

ENGINEERING TRIPOS PART IIA

Tuesday 27 April 2010 9 to 10.30

Module 3F1

SIGNALS AND SYSTEMS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Version: Final

1 (a) The open–loop system has two poles at $-0.5 \pm j0.8062$. The pole-zero diagram in Fig. 4 shows the poles inside the unit disk, and so the system is stable. [20%]

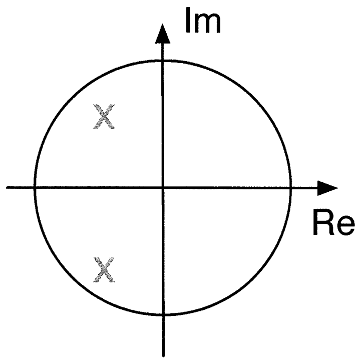


Fig. 4

(b) The closed–loop transfer function is given by

$$T(z) = \frac{Y(z)}{R(z)} = \frac{K(z)P(z)}{1+K(z)P(z)} = \frac{K(z)}{z^2+z+0.9+K(z)}$$

[10%]

(c) The closed loop poles are the roots of $z^2+z+0.9+k=0$ which are given by

$$z = \frac{-1 \pm \sqrt{-4k-2.6}}{2}$$

For closed–loop stability we need the poles inside of the unit disk. For $-4k-2.6<0$ the poles are complex and therefore we need

$$\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{-4k-2.6}}{2}\right)^2 < 1$$

or $|-4k-2.6|<3$. Since $4k+2.6>0$, we get $2.6+4k<3$, or $-0.65<k<0.1$. For $-4k-7 \geq 0$ the poles are real and therefore we need

$$-1 < \frac{-1 \pm \sqrt{-4k-2.6}}{2} < 1$$

or $-2 < -1 - \sqrt{-4k-2.6}$ and $-1 + \sqrt{-4k2.67} < 2$, which is equivalent to $\sqrt{-4k-2.6}<1$ and $\sqrt{-4k-2.6}<3$. Since $-4k-2.6 \geq 0$, we get $-0.9<k\leq -0.65$. Combining both, we get the final answer of $-0.9<k<0.1$. [20%]

(d) When $k \notin (-0.9, 0.1)$ the system is unstable and therefore y_n will grow unbounded. When $k \in (-0.9, 0.1)$, the closed–loop system is stable and we use the final value theorem (the closed–loop transfer function $T(z)$ was found in part (b)):

$$\lim_{n\to\infty} y_n = \lim_{z\to 1}(z-1)\frac{k}{z^2+z+0.9+k}\frac{z}{z-1} = \frac{k}{2.9+k}$$

[20%]

(e) At $z=-1$, $K(1)P(1) = -0.0115$ and therefore the phase must be -180 degrees. Plot D cannot be since the phase at frequency π *rad* is -360. Evaluating $K(z)P(z)$ at $z = e^{j2.1}$ gives $|KP| = 1.6$ and so, C cannot be. Plot B has a resonant peak at around frequency 1 *rad*. However, the resonance peak should be near frequency 2.1 *rad*, which is the angle of one of the complex poles in P. Thus, the correct one is A. [30%]

2 (a) (i) All elements are linear and time-invariant. [10%]

(ii) Consider the following input:

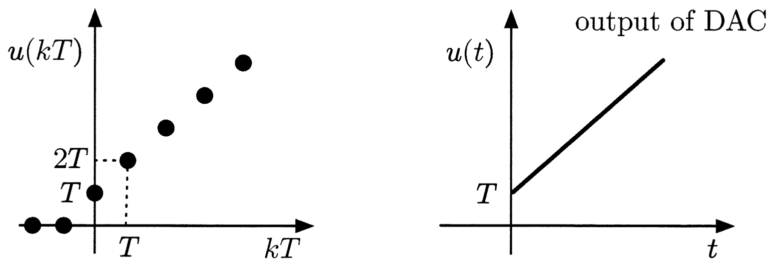


Fig. 5

This gives a convenient signal $u(t)$ and hence a useful formula for $H(z)$.

$$u(t) = T + t \quad \Rightarrow \quad U(s) = \frac{T}{s} + \frac{1}{s^2} = \frac{Ts+1}{s^2}$$

$$u(kT) = T + kT \quad \Rightarrow \quad U(z) = \frac{T}{1-z^{-1}} + \frac{Tz^{-1}}{(1-z^{-1})^2} = \frac{T}{(1-z^{-1})^2} = \frac{Tz^2}{(z-1)^2}$$

Then,

$$\{y(kT)\} = \mathcal{L}^{-1}\left(G(s)\frac{Ts+1}{s^2}\right)_{t=kT}$$

and the result now follows from $H(z) = Y(z)/U(z)$. [40%]

2 (b) (i) If $Y = g(X)$ & $y = g(x)$

the cdfs of X & Y are related by

$F_Y(y) = \Pr\{Y \le y\} = \Pr\{g(X) \le g(x)\}$

$= \Pr\{X \le x\}$ since $g(\cdot)$ is monotonically increasing

$= F_X(x)$

Hence the pdfs are related by

$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(x) = \frac{d}{dx} F_X(x) \cdot \frac{dx}{dy}$

$= f_X(x) \Big/ \left(\frac{dy}{dx}\right) = f_X(x) \Big/ g'(x)$

(ii) ~~Substituting~~ Substituting $f_X(x)$ & $f_Y(y)$ as given means that

$\frac{y}{\sigma^2} \exp\left(\frac{-y^2}{2\sigma^2}\right) \cdot g'(x) = \frac{1}{\alpha} \exp\left(\frac{-x}{\alpha}\right)$

Using the hint, try $g(x) = a x^b$

$\therefore g'(x) = a b x^{b-1}$

Substituting $y = g(x)$ into the LHS above gives:

$\frac{a x^b}{\sigma^2} \exp\left(\frac{-a^2 x^{2b}}{2\sigma^2}\right) \cdot a b x^{b-1}$

$= \frac{a^2 b}{\sigma^2} \cdot x^{2b-1} \exp\left(\frac{-a^2 x^{2b}}{2\sigma^2}\right)$

2(b)(ii)(cont)

For this to be equivalent to $\frac{1}{\lambda}\exp\left(\frac{-x}{\lambda}\right)$

we require that $x^{2b-1} = \text{const}$ & $x^{2b} = x$

Hence $b = \frac{1}{2}$

We also require that $\frac{1}{\lambda} = \frac{a^2 b}{\sigma^2} = \frac{a^2}{2\sigma^2}$

Hence $\cancel{a = \sqrt{\frac{2\sigma^2}{\lambda}}}$ $a^2 = \frac{2\sigma^2}{\lambda}$ & $\cancel{a = \sqrt{\frac{2\sigma^2}{\lambda}}}$

$\therefore a = \sigma\sqrt{\frac{2}{\lambda}}$ since $g(\cdot)$ is increasing.

$\therefore g(X) = aX^b = \sigma\sqrt{\frac{2}{\lambda}} \cdot X^{\frac{1}{2}} = \underline{\underline{\sigma\sqrt{\frac{2X}{\lambda}}}}$

3. (a) A random process $x(t)$ is wide-sense stationary if its mean value $E\{x(t)\}$, is independent of t, and if its auto-correlation function $E[x(t_1)\,x(t_2)]$ is a function only of $\tau = t_2 - t_1$ for all values of t_1 + t_2.

(b) The value of the autocorrelation function at $\tau = 0$ is $E[x(t_1).\,x(t_1)] = E[x^2(t_1)]$ for any t_1. Hence it is the mean squared value of $x(t)$, and will equal Q_x for the ACF given.

For a process with a uniform pdf $f_x(x) = \frac{1}{2A}$ from $-A$ to A, the mean squared value is

$$E[x^2(t)] = \int_{-\infty}^{\infty} x^2.\, f_x(x)\; dx$$

$$= \int_{-A}^{A} x^2.\, \frac{1}{2A}\, dx \qquad \left(\begin{array}{c}\text{area under pdf}\\ \text{must be unity.}\end{array}\right)$$

$$= \frac{1}{2A}.\left[\frac{x^3}{3}\right]_{-A}^{A} = \frac{1}{2A}\left(\frac{A^3}{3} + \frac{A^3}{3}\right) = \frac{A^2}{3}$$

$$\therefore\; Q_x = E[x^2(t)] = \underline{\underline{\frac{A^2}{3}.}}$$

3. (c). $y(t) = x(t) * h(t)$

$$= x(t) * \delta(t) - x(t) * \delta(t-T)$$
$$= x(t) - x(t-T)$$

$$r_{yy}(\tau) = E\left[y(t) \cdot y(t-\tau)\right]$$
$$= E\left[(x(t) - x(t-T)) \cdot (x(t-\tau) - x(t-\tau-T))\right]$$
$$= E\left[x(t)x(t-\tau)\right] - E\left[x(t-T)x(t-\tau)\right] - E\left[x(t)\,x(t-\tau-T)\right] + E\left[x(t-T)\,x(t-\tau-T)\right]$$
$$= r_{xx}(\tau) - r_{xx}(\tau-T) - r_{xx}(\tau+T) + r_{xx}(\tau)$$
$$= 2\,r_{xx}(\tau) - r_{xx}(\tau-T) - r_{xx}(\tau+T)$$

(d) The power spectrum $S_y(\omega)$ is the Fourier Transform of $r_{yy}(\tau)$ & $S_x(\omega)$ is the FT of $r_{xx}(\tau)$.

Hence $S_y(\omega) = 2S_x(\omega) - e^{-j\omega T} \cdot S_x(\omega) - e^{j\omega T} \cdot S_x(\omega)$

$$= S_x(\omega)\left(2 - (e^{j\omega T} + e^{-j\omega T})\right) = 2\,S_x(\omega)\,(1 - \cos\omega T)$$

$$= \underline{\underline{2\,Q_x T \operatorname{sinc}^2\left(\frac{\omega T}{2}\right) \cdot (1 - \cos\omega T)}}$$

(from E+I data book)

4. a) An N^{th} order Markov source is one in which the probability distribution of the source at time n, depends only on the previous N outputs from the source. Hence a 1^{st}-order Markov source depends only on the most recent one ~~previous~~ output from the source.

b) For the given source

$$P(S_n) = \sum_{k=1}^{3} P(S_n / S_{n-1}=k) \cdot P(S_{n-1}=k)$$

At equilibrium $P(S_n) = P(S_{n-1}) = P_e$.

If $P(S_{n-1}) = \left[\frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3}\right]$, then

$$P(S_n) = \left[0.1 \quad 0.7 \quad 0.2\right] \cdot \tfrac{1}{3} + \left[0.2 \quad 0.1 \quad 0.7\right] \cdot \tfrac{1}{3} + \left[0.7 \quad 0.2 \quad 0.1\right] \cdot \tfrac{1}{3}$$

$$= \left[1 \quad 1 \quad 1\right] \cdot \tfrac{1}{3} = \left[\tfrac{1}{3} \quad \tfrac{1}{3} \quad \tfrac{1}{3}\right].$$

Hence this is the equilibrium probability.

4. b) (cont.)

Mutual Information = $I(S_n, S_{n-1})$

$$= H(S_n) - H(S_n | S_{n-1})$$

Now $H(S_n) = -\sum_{k=1}^{3} \frac{1}{3} \log_2\left(\frac{1}{3}\right) = \log_2(3) = 1.5850$ bits

$$H(S_n | S_{n-1}) = H(S_n | S_{n-1} = A) \cdot P(S_{n-1} = A)$$
$$+ H(S_n | S_{n-1} = B) \cdot P(S_{n-1} = B)$$
$$+ H(S_n | S_{n-1} = C) \cdot P(S_{n-1} = C)$$
$$= \frac{3}{3} \cdot H([0.7\ 0.2\ 0.1])$$
$$= -0.7 \log_2(0.7) - 0.2 \log_2(0.2) - 0.1 \log_2(0.1)$$
$$= 1.1568 \text{ bits}$$

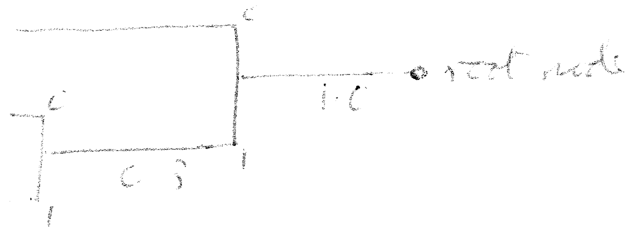
Hence $I(S_n ; S_{n-1}) = 1.5850 - 1.1568 = \underline{\underline{0.4282}}$ bits

7 [illegible]

4

c) Huffman code for one column of the table is

Code	Prob
0	0.7
10	0.2
11	0.1



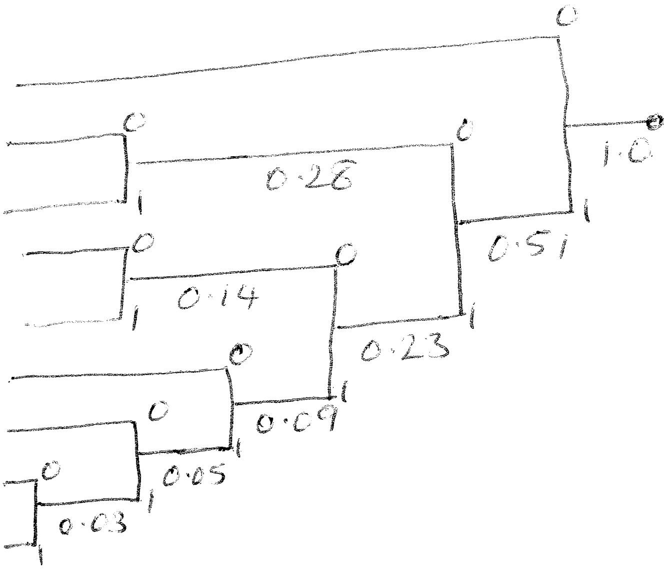
Average length $L = \sum_{i=1}^{3} p_i L_i$

$= 0.7.1 + 0.2.2 + 0.1.2 = \underline{1.3 \text{ bit/sym.}}$

All columns are equally likely to be used & use the same set of probabilities (just in different order) so $\underline{L = 1.3 \text{ bit/sym}}$ as an overall average

d) Given S_{n-1}, the possible options for (S_n, S_{n+1}) are the following 9 probabilities in decreasing order:

0	0.7^2	$= 0.49$
100	$0.7.0.2$	$= 0.14$
101	$0.2.0.7$	$= 0.14$
1100	$0.7.0.1$	$= 0.07$
1101	$0.1.0.7$	$= 0.07$
1110	0.2^2	$= 0.04$
11110	$0.2.0.1$	$= 0.02$
111110	$0.1.0.2$	$= 0.02$
111111	0.1^2	$= 0.01$



$\therefore L = 0.49.1 + 0.14(3+3) + 0.07(4+4) + 0.04.4$
$+ 0.02(5+6) + 0.01.6 = 2.33 \text{ bits / 2 sym}$
$= \underline{1.165} \text{ bit/sym.}$

4 (d) cont)

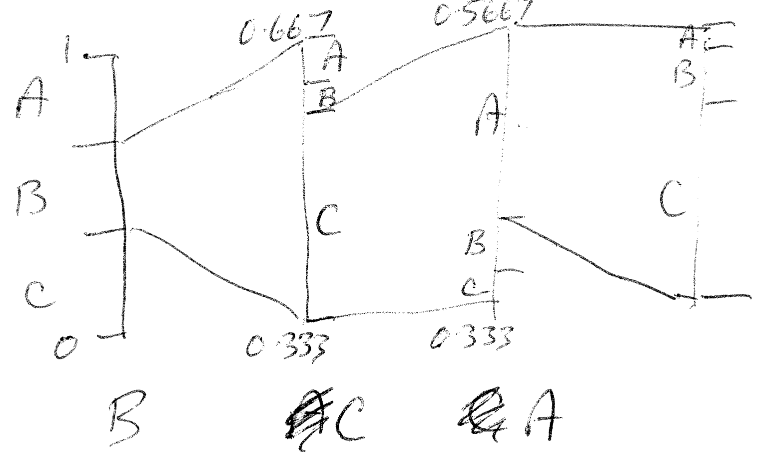
$\therefore$ Efficiency of order-2 method $= \dfrac{H(X_i|X_{i-1})}{L}$

$$= \frac{1.1568}{1.165} = 99.3\%$$

Efficiency of order-1 ~~a~~ method $= \dfrac{1.1568}{1.3} = 89.0\%$

Hence the order-2 method is much closer to 100% efficiency + probably worth the extra complexity of the 9-state Huffman code.

4(e) Arithmetic coding is a way of extending the source coding to order-N where N can be quite large. For Markov sources, Arithmetic coding allows us to easily change the probabilities expected for each new symbol ~~[illegible]~~ according to which symbol was transmitted before it. Hence for large N, the efficiency can approach arbitrarily close to 100%. E.g. for a message ~~[illegible]~~ BCA



Module 3F1, April 2009 – SIGNALS AND SYSTEMS – Answers

1 (a) Open-loop poles at $-0.5 \pm j0.8062$. Stable.
(b) $T(z) = K(z)/(z^2 + z + 0.9 + K(z))$.
(c) $-0.9 < k < 0.1$.
(d) For $-0.9 < k < 0.1$, $\lim_{n\to\infty} y_n = k/(2.9 + k)$.
(e) A.

2 (b) (ii) $g(x) = \sigma\sqrt{2/d}$.

3 (b) $Q_X = \frac{A^2}{3}$.
(d) $S_Y(\omega) = -4Q_X T \mathrm{sinc}^2(\omega T/2) - 4Q_X T \mathrm{sinc}^2(\omega T)$.

4 (b) $I(S_n; S_{n-1}) = 0.4282$ bits.
(c) $L = 1.3$ bits/sym.
(d) $L = 1.165$ bits/sym.

Version: Final