3F2 Systems and Control: 2010 Solutions

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1. (a) To check open-loop stability, evaluate the characteristic polynomial:

$$\begin{aligned}
\det(\lambda I - A) &= \begin{vmatrix} \lambda+4 & -2 & 0 \\ -2 & \lambda+6 & -2 \\ 0 & -2 & \lambda+4 \end{vmatrix} && (1)\\
&= (\lambda+4)\begin{vmatrix} \lambda+6 & -2 \\ -2 & \lambda+4 \end{vmatrix} - 4(\lambda+4) && (2)\\
&= (\lambda+4)[(\lambda+6)(\lambda+4) - 4 - 4] && (3)\\
&= (\lambda+4)(\lambda^2+10\lambda+16) && (4)\\
&= (\lambda+4)(\lambda+2)(\lambda+8) && (5)
\end{aligned}$$

In the last step the quadratic factor has been factorised by inspection, but that is not necessary. Already from (4) it can be seen that all the roots are in the left half-plane — or, less efficiently, the cubic polynomial can be multiplied out and the Routh-Hurwitz criterion then applied.

The roots all have negative real parts, so the system is open-loop stable.

(b) We have $\dot{x} = Ax + Bu$, and to hold temperatures constant we need $\dot{x} = 0$. So we need

$$0 = Ax_\infty + Bu_\infty \quad \text{and} \quad x_\infty = [25, 25, 25]^T. \tag{6}$$

Hence

$$Bu_\infty = -\begin{bmatrix} -4 & 2 & 0 \\ 2 & -6 & 2 \\ 0 & 2 & -4 \end{bmatrix}\begin{bmatrix} 25 \\ 25 \\ 25 \end{bmatrix} = \begin{bmatrix} 50 \\ 50 \\ 50 \end{bmatrix} \tag{7}$$

Now B is diagonal so very easy to invert, and we get

$$u_\infty = [p_1, p_2, p_3]^T = [10, 25, 25]^T \quad \text{(watts, if the units are standard).} \tag{8}$$

(c) Suppose that $Av = \lambda v$ for some eigenvalue λ.
By definition we have

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \cdots \tag{9}$$

so

$$\begin{aligned}
e^{At}v &= \left[I + tA + \frac{t^2}{2!}A^2 + \cdots\right]v && (10)\\
&= v + t\lambda v + \frac{t^2}{2!}\lambda^2 v + \cdots && (11)\\
&= \left(1 + t\lambda + \frac{t^2\lambda^2}{2!} + \cdots\right)v && (12)\\
&= e^{\lambda t}v && (13)
\end{aligned}$$

Hence v is an eigenvector of e^{At} (with eigenvalue $e^{\lambda t}$).

Some candidates worked with $A = W\Lambda W^{-1}$, where W is the matrix of eigenvectors. This is ok, except that W^{-1} does not always exist, so the above solution is better.

(d) To check that v is an eigenvector of A, evaluate Av and check that it is of the form λv:

$$\begin{bmatrix} -4 & 2 & 0 \\ 2 & -6 & 2 \\ 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{14}$$

so $[1,1,1]^T$ *is* an eigenvector of A, with eigenvalue -2.
Now if the initial wafer temperature (above ambient) is $x(0)$ then, with $u = 0$, its temperature will evolve as

$$x(t) = e^{At}x(0) \tag{15}$$

If its initial temperature is uniform then $x(0) = \theta_0[1,1,1]^T$, where θ_0 is the initial temperature of each segment. But $[1,1,1]^T$ is an eigenvector of A, hence also of e^{At}, by part (c).
Hence

$$x(t) = \theta_0 e^{-2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{16}$$

(e) Now $u = -Kx$, so $\dot{x} = (A - BK)x$. But $BK = cI$, so $\dot{x} = (A - cI)x$, and $x(t) = e^{(A-cI)t}x(0)$. But $(A - cI)v = Av - cv = (\lambda - c)v$ if v is an eigenvector of A with eigenvalue λ. Hence $[1,1,1]^T$ is an eigenvector of $A - cI$, and so, by the same argument as in part (d), $x(t)$ will remain uniform under the state feedback $u = -Kx$ if $BK = cI$ and $x(0) = \theta_0[1,1,1]^T$. (In fact the closed-loop evolution will be $x(t) = \theta_0 e^{(-2-c)t}[1,1,1]^T$.)
Some candidates checked that v is an eigenvector for the specific $A - cI$ matrix that they had, which is also ok.

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2. (a) Every point on the real axis is to the left of either 0 or 4 poles (and there are no zeros). Hence no point on the real axis is on the root locus.
Number of poles: $n = 4$. Number of zeros: $m = 0$. Hence there are $n - m = 4$ asymptotes.
Angles of asymptotes to positive real axis are $(2k+1)\pi/4$, namely $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.
Point of intersection of asymptotes with real axis is $(-8+0)/(4-0) = -2$.
So the asymptotes are as shown in Fig.1.
In fact, each asymptote *is* a branch of the root locus, as can be seen from the following argument. Consider an arbitrary point on one of the asymptotes, such as $s_0 = -1 + j$. Then $\arg G_1(s_0) = -4\arg(s_0 + 2) = -4 \times (\pi/4) = -\pi$. Since this is an odd multiple of π, s_0 is on the root locus. A similar argument holds for any point on each of the 4 asymptotes. Thus Fig.1 in fact shows the root-locus diagram, not just the asymptotes. Hence the root-locus diagram consists entirely of straight-line segments.
Note: The previous paragraph, about the asymptotes, is not a necessary part of the solution. One could jump straight to the final argument.

(b) A pair of poles $(s_0, \bar{s}_0)$ corresponds to damping factor $1/\sqrt{2}$ if $\arg s_0 = 5\pi/4$. Hence, from the root-locus diagram, we must have $s_0 = -1 + j$. Then

$$|G_1(s_0)| = 1/|s_0 + 2|^4 = 1/\sqrt{2}^4 = 1/4. \tag{17}$$

But for s_0 to be a closed-loop pole requires that $1 + k_1 G_1(s_0) = 0$, hence $k_1 = 1/|G_1(s_0)| = 4$.

(c) For $G_2(s)$: Every point on the real axis to the left of -4 is to the left of 5 poles and zeros. Hence every such point is on the root locus. Every point to the right of -4 is to the left of 0 or 4 poles and zeros, hence is not on the root locus.
The number of asymptotes is $n - m = 4 - 1 = 3$, and the zero at -4 attracts one branch of the root locus.
Angles of asymptotes to positive real axis are $(2k+1)\pi/3$, namely $\pi/3, \pi, 5\pi/3$.
Point of intersection of asymptotes with real axis is $(-8+4)/(4-1) = -4/3$.
Breakaway points are at values of s for which $G_2'(s) = 0$:

$$\frac{dG_2(s)}{ds} = \frac{(s+2)^4 - 4(s+4)(s+2)^3}{(s+2)^8} = 0 \tag{18}$$

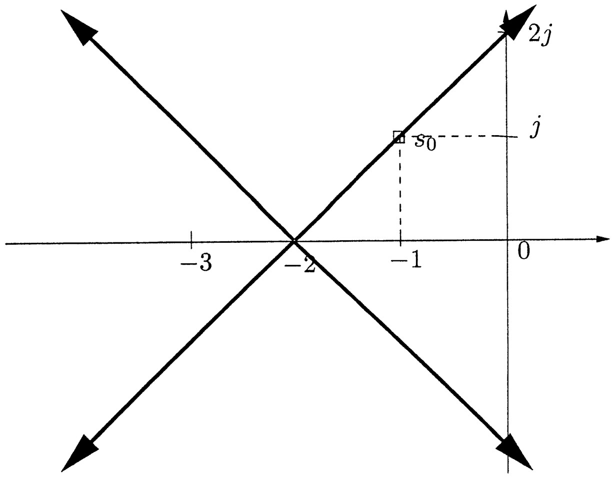

Figure 1: Root-locus diagram for Q.2(a)

when

$$(s+2)^3[(s+2)-4(s+4)] = -(s+2)^3(3s+14) = 0 \tag{19}$$

namely when $s=-2$ and $s=-14/3$.
Thus the root-locus diagram looks something like Fig.2.

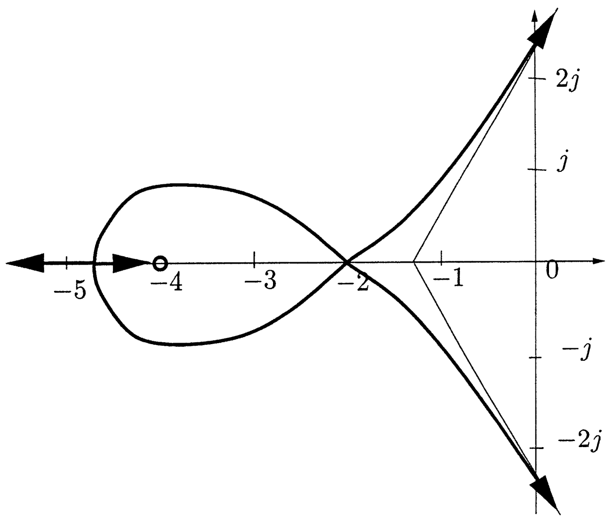

Figure 2: Root-locus diagram for Q.2(c)

(d) i. Let r be the reference input to the feedback loop, then the error is $\bar{e}(s) = \bar{r}(s) - L(s)\bar{e}(s)$. Hence

$$\bar{e}(s) = \frac{1}{1+L(s)}\bar{r}(s) \tag{20}$$

Now by the Final Value Theorem, and assuming $\bar{r}(s) = 1/s$, we have

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\bar{e}(s) = \lim_{s\to 0} s\frac{1}{1+L(s)}\frac{1}{s} = \frac{1}{1+L(0)} \tag{21}$$

(and $1+L(0) \neq 0$ since the closed loop is assumed to be (asymptotically) stable). (Alternatively, an argument about finding the frequency response at frequency 0 can be used to get the same result.)
In most practical cases the feedback design is such that $L(0) \gg 1$, so $1+L(0) \approx L(0)$, and hence

$$\lim_{t\to\infty} e(t) \approx \frac{1}{L(0)} \tag{22}$$

ii. In the case $L(s) = k_1 G_1(s)$ we have $L(0) = 4/16 = 1/4 < 1$, so the approximation cannot be used in this case, and we have

$$\lim_{t\to\infty} e(t) = \frac{1}{1+\frac{1}{4}} = \frac{4}{5} \tag{23}$$

Comment: In this case the closed loop is asymptotically stable and fairly well damped (damping factor $1/\sqrt{2}$ is a bit low, but commonly accepted in chemical process industries). However the steady-state error is very large (80%) which would be unacceptable in most applications. Furthermore the gain $|L(j\omega)|$ decreases monotonically with ω (as can be seen by considering the Bode plot of $G_1(s)$), so is low at all frequencies. Hence very poor disturbance rejection will be obtained at all frequencies. The open loop is already stable, so the feedback is not required for stabilisation. In summary this is a very poor design and there would not be much point in having such a feedback system with these characteristics.

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3. (a) The controllability matrix is

$$P = [B, AB] = \begin{bmatrix} b_1 & a_{11}b_1 \\ 0 & b_1 \end{bmatrix} \tag{24}$$

which has rank = 2, since $b_1 \neq 0$. Thus the condition for controllability is satisfied.

(b) A state-feedback scheme is $\delta = -Kx$, where $K = [k_1, k_2]$. Substituting this into the equation $\dot{x} = Ax + B\delta$ gives $\dot{x} = (A - BK)x$. The closed-loop poles are the eigenvalues of $A - BK$. We have:

$$A - BK = \begin{bmatrix} a_{11} & a_{12} \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} b_1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} a_{11} - b_1k_1 & a_{12} - b_1k_2 \\ 1 & 0 \end{bmatrix} \tag{25}$$

Now we can proceed in one of two ways:
Either: Find the characteristic polynomial of $A - BK$:

$$\det(\lambda I - A + BK) = \begin{vmatrix} \lambda - a_{11} + b_1k_1 & -a_{12} + b_1k_2 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + \lambda(-a_{11} + b_1k_1) + (-a_{12} + b_1k_2) \tag{26}$$

But this should be the same as $(\lambda + 1)^2$. Hence

$$-a_{11} + b_1k_1 = 2 \quad \text{and} \quad -a_{12} + b_1k_2 = 1 \tag{27}$$

Or: Use the fact that the sum of eigenvalues is the trace, and the product of eigenvalues is the determinant:

$$\text{trace}(A - BK) = a_{11} - b_1k_1 = -2, \qquad \det(A - BK) = -a_{12} + b_1k_2 = 1 \tag{28}$$

In either case we get:

$$k_1 = \frac{a_{11} + 2}{b_1}, \qquad k_2 = \frac{a_{12} + 1}{b_1} \tag{29}$$

(c) Integral action may be obtained by introducing a third state variable which is the integral of the error: $\dot{x}_3 = \theta_d - \theta$ if θ_d is the demanded pitch angle (if $\theta_d = 0$, as in this case, then $\dot{x}_3 = -\theta$). The dynamics are now augmented as follows:

$$\begin{bmatrix} \ddot{\theta} \\ \dot{\theta} \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \theta \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix} \delta + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \theta_d \tag{30}$$

State feedback of the form $\delta = -Kx$ is still used, but K now has 3 elements, and there are 3 closed-loop poles to be placed.

If K is such that the closed loop is asymptotically stable then the system will converge to an equilibrium in which $\dot{x}_3 = 0$, and hence $\theta = \theta_d$. The corresponding equilibrium value of δ will be whatever is required for this to occur. For this scheme to be effective, the 'disturbances' due to centre of gravity changes, force imbalances etc, must be piecewise-constant with only occasional jumps, or changing very slowly in relation to the desired closed-loop dynamics.

(d) Proceeding as in part (b), substituting the given values into (30):

$$A - BK = \begin{bmatrix} -2 & 0.5 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} = \begin{bmatrix} -2-k_1 & 0.5-k_2 & -k_3 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad (31)$$

This has characteristic polynomial:

$$\begin{vmatrix} \lambda+2+k_1 & k_2-0.5 & k_3 \\ -1 & \lambda & 0 \\ 0 & 1 & \lambda \end{vmatrix} = (\lambda+2+k_1)\lambda^2 + 1[\lambda(k_2-0.5) - k_3] \quad (32)$$

$$= \lambda^3 + (2+k_1)\lambda^2 + (k_2-0.5)\lambda - k_3 \quad (33)$$

$$\equiv (\lambda+1)^3 \quad (34)$$

$$= \lambda^3 + 3\lambda^2 + 3\lambda + 1 \quad (35)$$

So comparing coefficients we have

$$k_1 = 1, \quad k_2 = 3.5, \quad k_3 = -1 \quad (36)$$

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4. (a) A system is observable if the initial condition $x(0)$ can be inferred from observations of u and y over some interval of time $0 \le t \le T$.

Several alternative meanings also acceptable, but some precision is needed. For example, if $x(t_0)$ is to be inferred then the time interval must include t_0.

A test for a linear system being observable is to test whether rank$(Q) = n$, where n is the state dimension and

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad (37)$$

(Alternatively, if the system is asymptotically stable, one can test whether the observability Gramian is non-singular.)

(b) If $u(t) = 0$ for all t then $y(t) = Ce^{At}x_0$. Therefore

$$\int_0^\infty \|y(t)\|^2 dt = \int_0^\infty y(t)^T y(t) dt = \int_0^\infty [Ce^{At}x_0]^T Ce^{At}x_0 dt$$

$$= x_0^T \left[\int_0^\infty e^{A^T t} C^T C e^{At} dt\right] x_0 = x_0^T W_o x_0 \quad (38)$$

(c) Let $x_1 = \theta$, $x_2 = \dot\theta$, and $x = [x_1, x_2]^T$. Then the given differential equation is equivalent to the standard linear system in state-space form, if

$$A = \begin{bmatrix} 0 & 1 \\ -\epsilon & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \sqrt{2\epsilon} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad (39)$$

Check of observability:

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (40)$$

which has rank $= 2$ always (independently of ϵ).

(d) i. Using the hint, and recalling that $de^{At}/dt = Ae^{At} = e^{At}A$, we have

$$\frac{d}{dt}\left[e^{A^T t} C^T C e^{At}\right] = A^T e^{A^T t} C^T C e^{At} + e^{A^T t} C^T C e^{At} A \quad (41)$$

Integrating both sides:

$$\left[e^{A^Tt}C^TCe^{At}\right]_0^\infty = \int_0^\infty \left[A^Te^{A^Tt}C^TCe^{At} + e^{A^Tt}C^TCe^{At}A\right] dt \tag{42}$$

$$= A^T \int_0^\infty e^{A^Tt}C^TCe^{At}dt + \int_0^\infty e^{A^Tt}C^TCe^{At}dtA \tag{43}$$

$$= A^TW_o + W_oA \tag{44}$$

But $\left[e^{A^Tt}C^TCe^{At}\right]_0^\infty = -C^TC$ follows from the fact that the observability Gramian is only defined for asymptotically stable systems, so $e^{At} \to 0$ as $t \to \infty$.

ii. Substituting the values of A and C into the given matrix equation, and noting that $w_{12} = w_{21}$ (since W_o is symmetric) gives

$$\begin{bmatrix} 0 & -\epsilon \\ 1 & -1 \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix} + \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -\epsilon & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \tag{45}$$

which is the same as

$$\begin{bmatrix} -\epsilon w_{12} & -\epsilon w_{22} \\ w_{11} - w_{12} & w_{12} - w_{22} \end{bmatrix} + \begin{bmatrix} -\epsilon w_{12} & w_{11} - w_{12} \\ -\epsilon w_{22} & w_{12} - w_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \tag{46}$$

or

$$\begin{bmatrix} -2\epsilon w_{12} & w_{11} - w_{12} - \epsilon w_{22} \\ w_{11} - w_{12} - \epsilon w_{22} & 2(w_{12} - w_{22}) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \tag{47}$$

Hence we have

$$w_{12} = \frac{1}{2\epsilon}, \quad w_{22} = w_{12} = \frac{1}{2\epsilon}, \quad w_{11} = w_{12} + \epsilon w_{22} = \frac{1}{2\epsilon} + \frac{1}{2} \tag{48}$$

or

$$W_o = \frac{1}{2\epsilon} \begin{bmatrix} 1+\epsilon & 1 \\ 1 & 1 \end{bmatrix} \tag{49}$$

3F2 Systems and Control: 2010 Numerical answers

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1. (b): $u = [10, 25, 25]^T$. (d): $x(t) = e^{-2t}\theta_0\nu$, where θ_0 is the initial temperature.
2. (b): $k_1 = 4$. (d)(ii): Steady-state error $= \frac{4}{5}r_0$ if r_0 is the amplitude of a step reference signal.
3. (b): $K = \left[\frac{a_{11}+2}{b_1}, \frac{a_{12}+1}{b_1}\right]$. (d): $K = [1, 3.5, -1]$.
4. (d)(ii): $W_o = \frac{1}{2\epsilon}\begin{bmatrix} 1+\epsilon & 1 \\ 1 & 1 \end{bmatrix}$.