ENGINEERING TRIPOS PART IIA

Thursday 22 April 2010 9.00 to 12.00

Module 3A1

FLUID MECHANICS I

*Answer not more than **five** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments:

- *3A1 Data Sheet for Applications to External Flows (2 pages);*
- *Boundary Layer Theory Data Card (1 page);*
- *Incompressible Flow Data Card (2 pages).*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Version: Final version

1 A particular body is to be modelled by a line source of strength $1/\sqrt{2}$ at the origin, a vortex of strength $-1/\sqrt{2}$ (clockwise) also at the origin and a uniform velocity from left to right of magnitude $1/2\pi$.

(a) Write down the complex potential for this flow. [10%]

(b) Find the position of the stagnation points. [10%]

(c) Sketch the streamline pattern. [30%]

(d) What is the width of the modelled body (in a direction perpendicular to the uniform flow) at a location far downstream (i.e. far to the right). Hint: consider the top streamline representing the body at large distance $\theta \to 0$ and the bottom streamline representing the body $\theta \to 2\pi$. [50%]

2 A line source of strength 0.5 is located at the point $A = (1,0)$ and a line sink of strength 1 at the origin as shown in the Fig. 1.

(a) Write down the complex potential for this flow. [10%]

(b) Find the stagnation points and sketch the flow pattern. [10%]

(c) If a non-diffusive toxic gas emanates from the line source, find the value of Y beyond which a person would be safe on the line $x = 1$ (i.e., not exposed to the gas). How far in the direction of positive x would the person be safe on the line $y = 0$? [30%]

(d) A uniform flow of magnitude U from left to right is added. What is the maximum magnitude of this flow such that there is a location downstream on the line $y = 0$ where a person would still be safe. [50%]

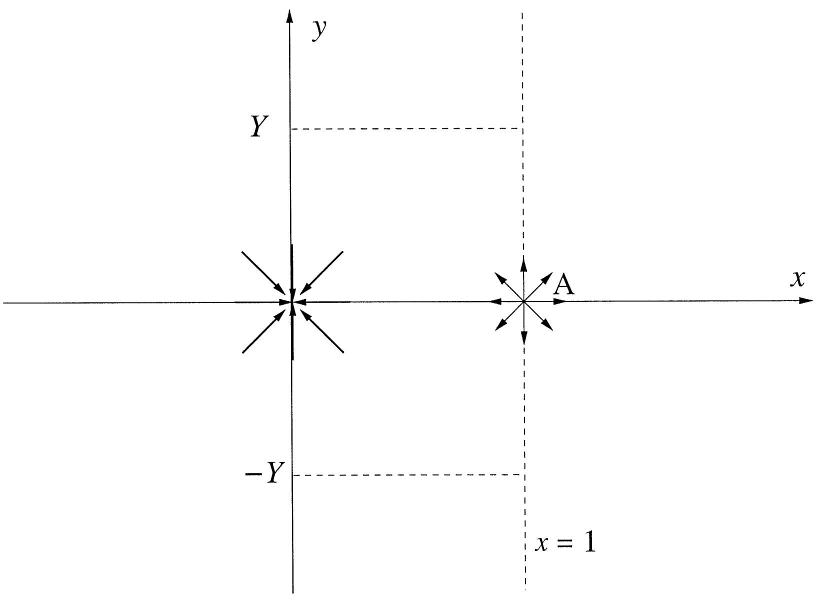


Fig. 1

3 A doublet of strength $8\pi U a^2$ is located at a distance a above an infinite plane wall in a uniform flow of velocity U from left to right as shown in the Fig. 2.

(a) Write down the complex potential for this flow. [10%]

(b) Find the positions of the stagnation points. [30%]

(c) This flow pattern may be used to represent a solid body which is symmetrical about the x-axis. Find the length of the body along the x-axis that is represented by the streamline passing through the stagnation points. [20%]

(d) Find the total width of this body normal to the flow at $x = 0$. [40%]

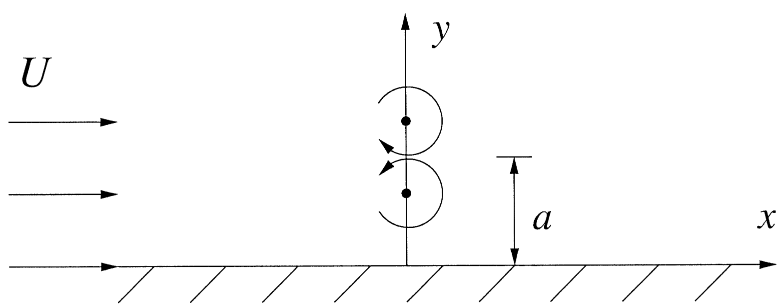


Fig. 2

4 (a) Consider a two-dimensional inviscid and incompressible flow. Show that

$$\omega = -\nabla^2 \psi$$

where ω is the vorticity and ψ is the streamfunction. [20%]

(b) A Rankine vortex can be modeled by

$$\begin{cases} \omega = \Omega & \text{for } r < a, \\ \omega = 0 & \text{for } r \geq a, \end{cases}$$

where Ω and a are constants.

Take $\psi = \psi(r)$ as the streamfunction for the Rankine vortex (because there is no θ variation). Find expressions of ψ for $r < a$, and $r \geq a$. There will be two undetermined constants for the inner flow and two undetermined constants for the outer flow. You may use $\nabla^2 \psi = \frac{1}{r}\frac{d}{dr}(r\frac{d\psi}{dr})$ as ψ is independent of θ. [40%]

(c) Determine the above constants by imposing the following boundary conditions: (i) $\psi = 0$ at $r = a$, (ii) the azimuthal velocity u_θ is continuous at $r = a$, and (iii) there is no singularity at $r = 0$. [30%]

(d) Comment on the relation of the Rankine vortex to a line vortex. [10%]

5 A laminar boundary layer on a flat plate has a velocity profile

$$\frac{U}{U_1} = \frac{4}{3}\eta - \frac{1}{3}\eta^4$$

where U_1 is the external velocity (far from the wall) and $\eta = y/\delta$ where δ is a measure of the boundary layer thickness.

(a) By consideration of the boundary layer equation very close to the wall, show that the pressure gradient must be zero. [10%]

(b) Find:

(i) The displacement thickness δ^*;

(ii) The momentum thickness θ;

(iii) The shape factor H;

(iv) The local skin friction coefficient $C_f^{\prime}$. [20%]

(c) Write down the momentum integral equation in terms of the quantities you found in part (b). [10%]

(d) Find the variation of $C_f^{\prime}$ in terms of $Re_x = U_1 x/\nu$, where x is the distance from the leading edge of the plate (where $\delta = 0$) and ν is the kinematic viscosity. [60%]

6 A turbulent boundary layer develops over a rough plate with a roughness height h. The mean velocity profile outside the roughness sublayer is

$$\frac{U}{U_\tau} = \frac{1}{\kappa}\ln\left(\frac{y}{h}\right) + C\left(\frac{hU_\tau}{\nu}\right)$$

where U is the streamwise velocity, κ the von Karman constant, U_τ the wall-shear velocity ($U_\tau = \sqrt{\tau_w/\rho}$, τ_w is the wall shear-stress and ρ the density) and ν is the kinematic viscosity of the fluid), and C is a function.

(a) Show that the mean velocity gradient is independent of the viscosity and the roughness height h. [30%]

(b) Find an expression for the variation of the local skin friction coefficient $C_f^{'} = \tau_w/(\frac{1}{2}\rho U_1^2)$ in terms of δ/h and hU_τ/ν, where U_1 is the velocity outside the boundary layer, and δ the boundary layer thickness. [30%]

(c) Show that the velocity defect $(U_1 - U)/U_\tau$ is independent of the roughness Reynolds number hU_τ/ν and also the height h. [20%]

(d) The roughness height is varied in the downstream direction such that it increases and is proportional to the boundary layer thickness: $h = a\delta$, where a is some constant. It is found that $C(hU_\tau/\nu) \to C_0$ as $hU_\tau/\nu \to \infty$. At this limit, find an expression for the skin friction coefficient $C_f^{'}$. [20%]

7 (a) Sketch a typical streamline just outside the boundary layer on the suction surface of an infinite swept wing, and explain its shape. Also show a streamline inside the boundary layer, and explain why it does not lie directly below the outer streamline. [35%]

(b) Figure 3 shows a finite, planar, swept wing of semi-span s.

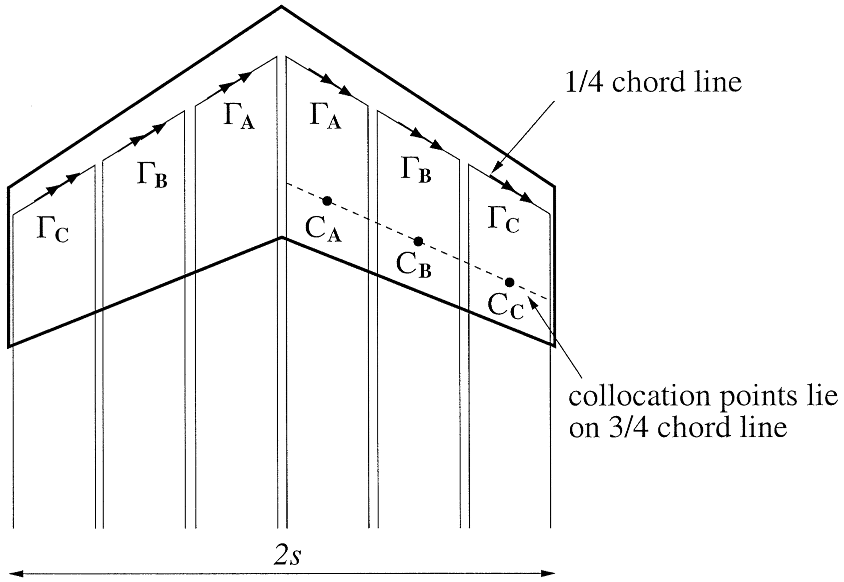


Fig. 3

(i) Comment on how you would expect the flow streamlines to compare with those on an infinite wing with the same sweep-back. [10%]

(ii) The horseshoe vortices shown in Fig. 3 represent an extended lifting-line model of the wing. The constant of proportionality specifying the downwash velocities, V_{dA} , V_{dB} and V_{dC} at the collocation points C_A , C_B and C_C due to the horseshoe vortices are given in table 1. At a certain incidence, α , the vortex circulations are found to be: $\Gamma_A = 0.1105sU$, $\Gamma_B = 0.1043sU$, $\Gamma_C = 0.0852sU$, where U is the free-stream velocity. Check that these values are consistent, and find α . [20%]

(iii) For an unswept wing with the same chord distribution and overall lift coefficient, the corresponding lifting-line model has $\Gamma_A = 0.1193sU$, $\Gamma_B = 0.1042sU$ and $\Gamma_C = 0.0765sU$. Comment on how sweep-back affects the spanwise lift distribution. [10%]

(c) On the unswept wing, stall is observed to initiate at midspan. Deduce, giving your reasoning, where stall will initiate on the swept wing and suggest measures that could be employed to alter this behaviour.

[25%]

Vortices	Collocation point downwash velocities		
	A	B	C
Pair A	$sV_{dA}/\Gamma_A = 1.6352$	$sV_{dB}/\Gamma_A = -0.7006$	$sV_{dC}/\Gamma_A = -0.1583$
Pair B	$sV_{dA}/\Gamma_B = -0.3710$	$sV_{dB}/\Gamma_B = 2.2888$	$sV_{dC}/\Gamma_B = -0.6089$
Pair C	$sV_{dA}/\Gamma_C = -0.0477$	$sV_{dB}/\Gamma_C = -0.2740$	$sV_{dC}/\Gamma_C = 2.5705$

Table 1

8 (a) A thin aerofoil of chord c has a camber-line given by

$$y_c = 2Y\frac{x}{c}\left(1-\frac{x}{c}\right)$$

where x is the chordwise distance from the leading edge and Y is a constant. Find the angle of attack for zero lift, and define the lumped-parameter model of the aerofoil. [30%]

(b) The wing-flap configuration shown in Fig. 4 consists of a cambered wing aerofoil with chord $4c_{flap}$ and zero-lift angle α_{0w}, and a symmetrical flap aerofoil deflected by an angle δ_{flap}. The oncoming flow is at incidence α. Estimate the lift coefficients of the wing and flap, based on their respective chords. (You may assume that the angles α, α_{0w} and δ_{flap} are all small). [40%]

(c) Comment on the constants of proportionality between the lift coefficients and the angles α, α_{0w} and δ_{flap}. [30%]

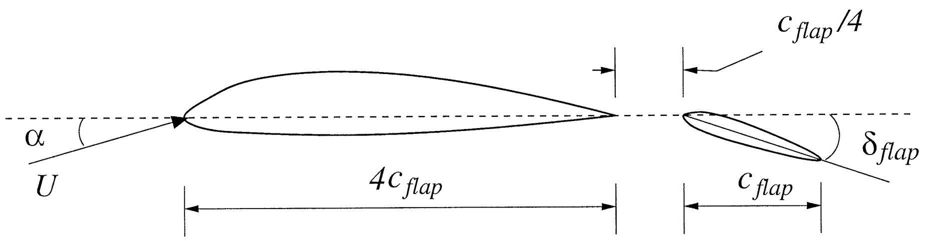


Fig. 4

END OF PAPER

3A1 Data Sheet for Applications to External Flows

Aerodynamic Coefficients

For a flow with free-stream density, ρ, velocity U and pressure p_∞:

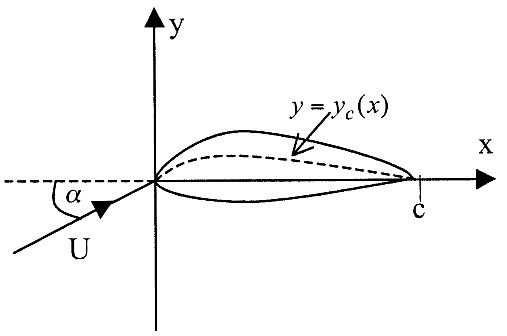
Pressure coefficient: $c_p = \dfrac{p - p_\infty}{\frac{1}{2}\rho U^2}$

Section lift and drag coefficients: $c_l = \dfrac{\text{lift}\,(N/m)}{\frac{1}{2}\rho U^2 c}$, $c_d = \dfrac{\text{drag}\,(N/m)}{\frac{1}{2}\rho U^2 c}$ (section chord c)

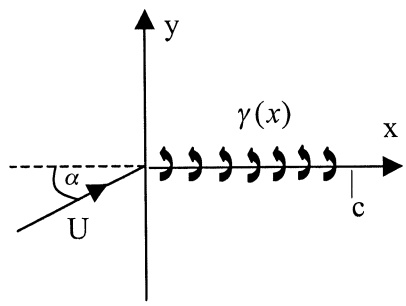
Wing lift and drag coefficients: $C_L = \dfrac{\text{lift}\,(N)}{\frac{1}{2}\rho U^2 S}$, $C_D = \dfrac{\text{drag}\,(N)}{\frac{1}{2}\rho U^2 S}$ (wing area S)

Thin Aerofoil Theory

Geometry



Approximate representation



Pressure coefficient: $c_p = \pm\gamma / U$

Pitching moment coefficient: $c_m = (\text{moment about } x = 0) / \frac{1}{2}\rho U^2 c^2$

Coordinate transformation: $x = c(1 + \cos\theta)/2 = c\cos^2(\theta/2)$

Incidence solution: $\gamma = -2U\alpha \dfrac{1 - \cos\theta}{\sin\theta}$, $c_l = 2\pi\alpha$, $c_m = c_l / 4$

Camber solution: $\gamma = -U\left[g_0 \dfrac{1 - \cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} g_n \sin n\theta\right]$, where

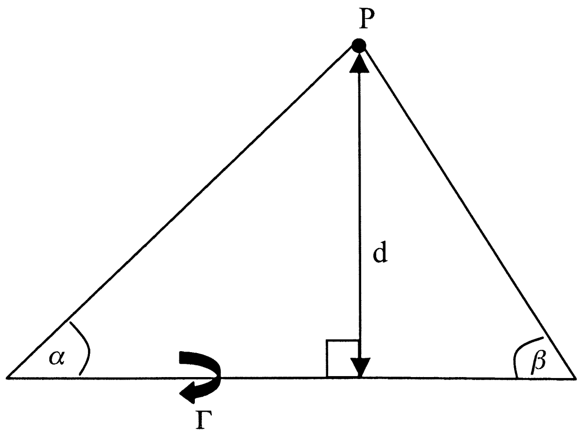
$$g_0 = \frac{1}{\pi}\int_0^\pi \left(-2\frac{dy_c}{dx}\right) d\theta, \quad g_n = \frac{2}{\pi}\int_0^\pi \left(-2\frac{dy_c}{dx}\right)\cos n\theta\, d\theta$$

$$c_l = \pi\left(g_0 + \frac{g_1}{2}\right), \quad c_m = \frac{\pi}{4}\left(g_0 + g_1 + \frac{g_2}{2}\right) = \frac{c_l}{4} + \frac{\pi}{8}(g_1 + g_2)$$

Glauert Integral

$$\int_{o}^{\pi} \frac{\cos n\phi}{\cos\phi - \cos\theta} d\phi = \pi \frac{\sin n\theta}{\sin\theta}$$

Line Vortices



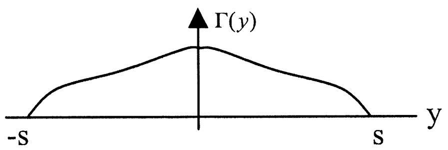
A straight element of circulation Γ induces a velocity at P of

$$\frac{\Gamma}{4\pi d}(\cos\alpha + \cos\beta)$$

perpendicular to the plane containing P and the element.

Lifting-Line Theory

Spanwise circulation distribution:



Aspect ratio: $A_R = 4s^2 / S$

Wing lift: $L = \rho U \int_{-s}^{s} \Gamma(y) dy$

Downwash angle: $\alpha_d(y) = \frac{1}{4\pi U} \int_{-s}^{s} \frac{d\Gamma(\eta)/d\eta}{y - \eta} d\eta$

Induced drag: $D_i = \rho U \int_{-s}^{s} \Gamma(y) \alpha_d(y) dy$

Fourier series for circulation: $\Gamma(y) = Us \sum_{\text{n odd}} G_n \sin n\theta$, with $y = -s\cos\theta$

Relation between lift and induced drag:

$$C_{Di} = (1 + \delta)\frac{C_L^2}{\pi A_R}, \text{ where } \delta = 3\left(\frac{G_3}{G_1}\right)^2 + 5\left(\frac{G_5}{G_1}\right)^2 + \ldots$$

Module 3A1
Boundary Layer Theory Data Card

Displacement thickness;

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_1}\right) dy$$

Momentum thickness;

$$\theta = \int_0^\infty \frac{(U_1 - u)u}{U_1^2} dy = \int_0^\infty \left(1 - \frac{u}{U_1}\right) \frac{u}{U_1} dy$$

Energy thickness;

$$\delta_E = \int_0^\infty \frac{(U_1^2 - u^2)u}{U_1^3} dy = \int_0^\infty \left(1 - \left(\frac{u}{U_1}\right)^2\right) \frac{u}{U_1} dy$$

$$H = \frac{\delta^*}{\theta}$$

Prandtl's boundary layer equations (laminar flow);

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{dp_1}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}$$

von Karman momentum integral equation;

$$\frac{d\theta}{dx} + \frac{H+2}{U_1}\theta\frac{dU_1}{dx} = \frac{\tau_o}{\rho U_1^2} = \frac{C_f'}{2}$$

Boundary layer equations for turbulent flow;

$$\begin{aligned} \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} &= \frac{-1}{\rho}\frac{d\overline{p}}{dx} - \frac{\partial \overline{u'v'}}{\partial y} + \nu\frac{\partial^2 \overline{u}}{\partial y^2} \\ \frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} &= 0 \end{aligned}$$

3A1 Incompressible Flow Data Card - Inviscid Flow

Continuity equation $\nabla \cdot \boldsymbol{u} = 0$

Momentum equation (inviscid) $\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla p + \rho \boldsymbol{g}$

D/Dt denote the material derivative, $\partial/\partial t + \boldsymbol{u} \cdot \nabla$

Vorticity $\boldsymbol{\omega} = \text{curl } \boldsymbol{u}$

Vorticity equation(Inviscid) $\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$

Kelvin's circulation theorem(Inviscid)

$$\frac{D\Gamma}{Dt} = 0, \qquad \Gamma = \oint \boldsymbol{u} \cdot \boldsymbol{dl} = \int \boldsymbol{\omega} \cdot \boldsymbol{dS}$$

For an irrotational flow

velocity potential (ϕ) $\boldsymbol{u} = \nabla\phi$ and $\nabla^2\phi = 0$

Bernouolli's equation for inviscid flow,

$$\frac{p}{\rho} + \frac{1}{2}V^2 + gz + \frac{\partial\phi}{\partial t} = \text{constant throughout the flowfield}, V = |\boldsymbol{u}|$$

TWO-DIMENSIONAL FLOW

Streamfunction(ψ)

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}$$

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad u_\theta = -\frac{\partial\psi}{\partial r}$$

Lift force Lift force/unit length $= \rho U(-\Gamma)$

Complex potential $F(z)$ for irrotational flows, with $z = x + iy$, $F(z) = \phi + i\psi$ and $dF/dz = u - iv$.

Examples of complex potentials

(i) uniform flow in x direction, $F(z) = Uz$

(ii) source at $z = z_o$, $F(z) = \frac{m}{2\pi}\ln(z - z_o)$

(iii)doublet at $z = z_o$,
with axis in the x direction $F(z) = \frac{\mu}{2\pi(z - z_o)}$

(iv)anticlockwise vortex
at $z = z_o$, $F(z) = -\frac{i\Gamma}{2\pi}\ln(z - z_o)$

TWO-DIMENSIONAL FLOW

Summary of simple 2-D flow fields

	ϕ	ψ	**circulation**	$\boldsymbol{u}$
Uniform flow	Ux	Uy	0	$u = U,\ v = 0$
Source at origin	$\frac{m}{2\pi}\ln r$	$\frac{m}{2\pi}\theta$	0	$u_r = \frac{m}{2\pi r},\ u_\theta = 0$
Doublet at origin θ is angle from doublet axis	$\frac{\mu\cos\theta}{2\pi r}$	$-\frac{\mu\sin\theta}{2\pi r}$	0	$u_r = -\frac{\mu\cos\theta}{2\pi r^2},\ u_\theta = -\frac{\mu\sin\theta}{2\pi r^2}$
Anticlockwise vortex at origin	$\frac{\Gamma}{2\pi}\theta$	$-\frac{\Gamma}{2\pi}\ln r$	Γ	$u_r = 0,\ u_\theta = \frac{\Gamma}{2\pi r}$

THREE-DIMENSIONAL FLOW

Summary of simple 3-D flow fields

	ϕ	$\boldsymbol{u}$
Source at origin	$-\frac{m}{4\pi r}$	$u_r = \frac{m}{4\pi r^2},\ u_\theta = 0,\ u_\phi = 0$
Doublet at origin θ is angle from doublet axis	$\frac{\mu\cos\theta}{4\pi r^2}$	$u_r = -\frac{\mu\cos\theta}{2\pi r^3},\ u_\theta = -\frac{\mu\sin\theta}{4\pi r^3},\ u_\phi = 0$

TBN 2009

Numerics Answers

Jie Li

Quesion 1,
(b) Stagnation point $z = (1+i)/\sqrt{2}$.
(d) Body width $\sqrt{2}\pi$.
Question 2,
(b) Stagnation point $z = 2$
(c) $y = \pm 1$. $x > 2$.
(d) Maximum velocity $U \approx 0.0139$.
Question 5,
(b) (iii) $H = 81/29$.
Question 7,
(b) (ii) $\alpha \approx 0.138$.