ENGINEERING TRIPOS PART IIA

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4 May 2010 9.00 to 10.30

Module 3C5

DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
3C5 Dynamics and 3C6 Vibration datasheet (6 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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1. A thin wire is wound to form a five-coil helical spring of radius R and pitch L as shown in Fig. 1(a). One full coil of mass m is cut from the spring as shown in Fig. 1(b). The shape of the single coil is described by the parametric equations

$$x = R\cos\theta, \quad y = R\sin\theta, \quad z = \frac{L\theta}{2\pi} \qquad \text{for } -\pi < \theta < \pi.$$

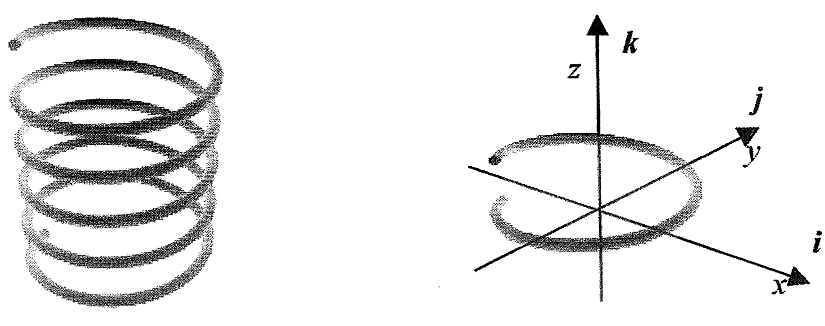
(a) Find the inertia matrix of the single coil at its centre of mass G. You may find it helpful to note that $\mathrm{d}m = (m/2\pi)\mathrm{d}\theta$. [50%]

(b) For the case $L = 2R$ show that the inertia matrix of a single coil is

$$mR^2\begin{bmatrix} \frac{5}{6} & 0 & 0 \\ 0 & \frac{5}{6} & -\frac{1}{\pi} \\ 0 & -\frac{1}{\pi} & 1 \end{bmatrix}.$$ [10%]

(c) Use the parallel axes theorem to find the inertia matrix for a five-coil helical spring of mass $5m$, length $10R$ and radius R. [20%]

(d) Find the magnitude of the couple required to maintain the five-coil spring spinning at steady angular velocity $\Omega \boldsymbol{k}$. [20%]



(a) (b)

Fig. 1

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2. A gyropendulum comprises a rotor of mass m at the end of a light rigid shaft OG of length L as shown in Fig. 2. The centre of mass of the rotor is at G and the axis of rotation is aligned with OG. The end of the shaft at O is fixed to a frictionless spherical joint. The principal moments of inertia for the gyropendulum at O (not G) are A, A, and C. The rotor spins with angular velocity ω and the angle between the shaft and the upward vertical is θ.

(a) For fast spin find the rate of steady precession for any angle θ. [30%]

(b) For $\theta < \pi/2$ find the rate of spin ω below which steady precession is impossible. How does this relate to the stability of a spinning top? [30%]

(c) For $\theta > \pi/2$ show that steady precession is always possible. [10%]

(d) For θ close to π and for very slow spin show that there are two possible solutions for steady precession. Explain how these two solutions relate to the small-amplitude frequency $\sqrt{g/L}$ of a simple conical pendulum of length L. [30%]

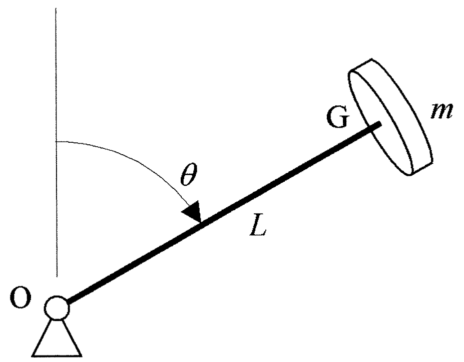

Fig. 2

3. A solid billiard ball of mass m and radius a is free to roll without slip on a flat horizontal table. The table is rotating at a steady angular velocity Ω about a vertical axis through O. In a particular steady-state motion the ball is moving with speed V on a circle of radius R centred on O as shown in Fig. 3.

(a) Use a no-slip condition to show that the magnitude of the steady angular velocity ω of the ball is $(R\Omega - V)/a$. Show on a diagram the direction of this angular velocity. [20%]

(b) On a free-body diagram of the ball show all forces and hence deduce the steady couple acting on the ball. [30%]

(c) Use gyroscope equations or otherwise to show that $V = \alpha R\Omega$ and find the value of the constant α. [50%]

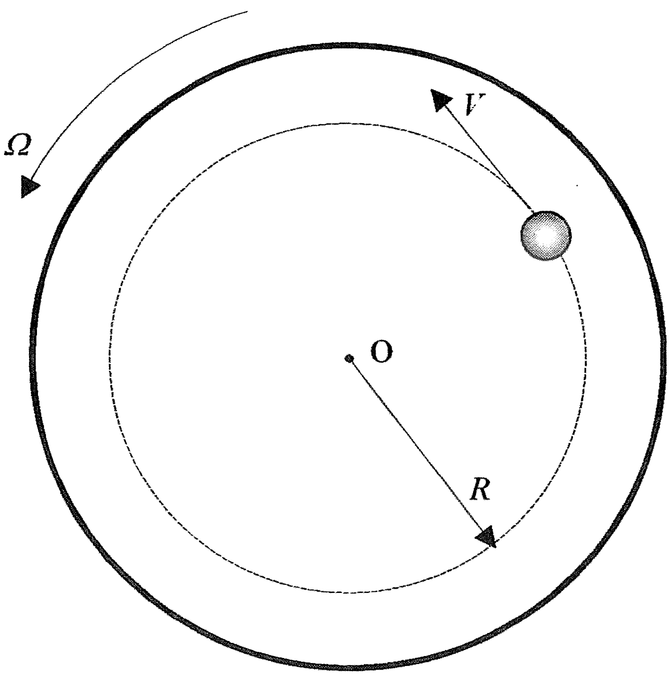

Fig.3

4. The two degree of freedom system shown in Fig. 4 consists of a body of mass m_1 which moves in the horizontal direction, and a body of mass m_2 which moves freely along an inclined surface of the first body. The second body is connected to a spring of stiffness k, as shown in the figure, and a force P is applied to this body in the direction of the inclined surface, which is at an angle θ to the horizontal. The horizontal displacement of the first body is labelled x_1, and the sliding displacement of the second body is labelled x_2 (when $x_2 = 0$ the spring carries no force).

(a) By using Lagrange's equation, derive the equations of motion of the system, including the effects of gravity. Express these equations in matrix form and confirm that the mass matrix is given by

$$\begin{pmatrix} m_1 + m_2 & m_2 \cos\theta \\ m_2 \cos\theta & m_2 \end{pmatrix}$$ [40%]

(b) Calculate the natural frequencies and mode shapes associated with free vibration of the system. [40%]

(c) Derive expressions for the generalised momenta associated with the two generalised coordinates x_1 and x_2. One of the generalised momenta is conserved during free vibration: state which one, and explain this result in physical terms. [20%]

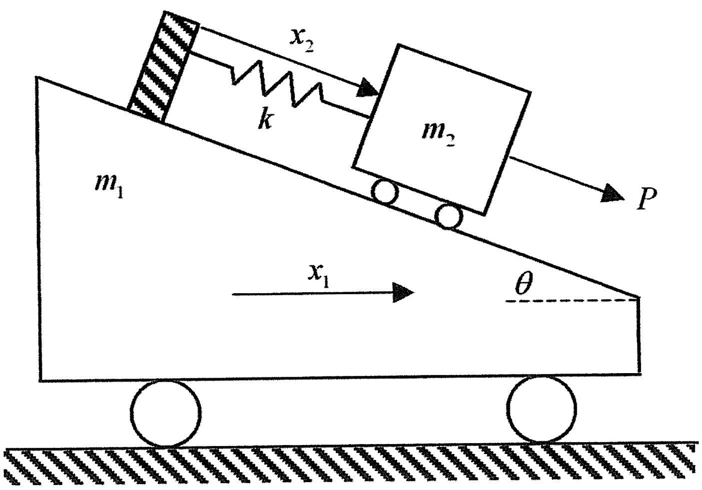

Fig. 4

5. (a) Explain how the effects of gravity can be included in Lagrange's equation in two ways: by including the gravitational potential energy, or by employing appropriate generalised forces. Demonstrate the equivalence of these two approaches for:

(i) a mass on a spring; [15%]

(ii) a simple pendulum. [20%]

(b) In the presence of damping the standard form of Lagrange's equation is modified to become

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}+\frac{\partial V}{\partial q_j}+\frac{\partial F}{\partial \dot{q}_j}=Q_j$$

where F is known as Rayleigh's dissipation function, and the other symbols have their usual meaning. For a simple mass/spring/damper system, the dissipation function has the form

$$F=\frac{\lambda \dot{x}^2}{2}$$

where λ is the damper rate and $\dot{x}$ is the velocity of the mass. Show that the use of the modified form of Lagrange's equation leads to the correct equation of motion for this system. [15%]

(c) The equation of motion of an N-degree-of-freedom linear system has the form

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{C}\dot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\mathbf{Q}$$

where **M**, **C** and **K** are respectively the mass, damping and stiffness matrices, **q** is the vector of generalised coordinates, and **Q** is the vector of generalised forces.

(i) Write down expressions for the kinetic and potential energies of the system, and show how in the absence of damping (**C**=**0**) the standard form of Lagrange's equation leads to the above equation of motion. [20%]

(ii) On the assumption that the matrix **C** is symmetric, deduce the form of the dissipation function for this system. [20%]

(iii) Discuss whether it is possible to derive a dissipation function if **C** is not symmetric. [10%]

END OF PAPER

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3C5 Dynamics: Answers to Tripos Paper 2010

1. (a) $\mathbf{I} = \frac{m}{12}\begin{pmatrix} 6R^2 + L^2 & 0 & 0 \\ 0 & 6R^2 + L^2 & -6RL/\pi \\ 0 & -6RL/\pi & 12R^2 \end{pmatrix}$.

   (b) $\mathbf{I} = \frac{5mR^2}{6}\begin{pmatrix} 53 & 0 & 0 \\ 0 & 53 & -6/\pi \\ 0 & -6/\pi & 6 \end{pmatrix}$.

   (c) $\mathbf{Q}=(5mR^2\Omega^2/\pi)\mathbf{i}$.

2. (a) $\dot{\phi} = mgL/(C\omega_3)$.

   (b) Stable for $\omega_3^2 > (4AmgL/C^2)\cos\theta$.

   (c) Always stable because $\cos\theta < 0$.

   (d) For $A \approx mL^2$ the two solutions are $\dot{\phi} = \frac{C\omega_3}{2A} \pm \sqrt{\frac{g}{L}}$.

3. (b) $Q = maV^2/R$ in the direction of the velocity V.

   (c) $\alpha = 2/7$.

4. (a) $\begin{pmatrix} m_1 + m_2 & m_2\cos\theta \\ m_2\cos\theta & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} P\cos\theta \\ P + m_2 g\sin\theta \end{pmatrix}$.

   (b) $\omega_1 = 0,\ \mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix};\ \omega_2^2 = \frac{k(m_1+m_2)}{m_2(m_1+m_2\sin^2\theta)},\ \mathbf{u}_2 = \begin{pmatrix} \frac{-m_2\cos\theta}{m_1+m_2} \\ 1 \end{pmatrix}$.

   (c) $p_1 = (m_1 + m_2)\dot{x}_1 + m_2\dot{x}_2\cos\theta$ (conserved)
   $p_2 = m_2\dot{x}_2 + m_2\dot{x}_1\cos\theta$ (not conserved).

5. (b) $M\ddot{x} + \lambda\dot{x} + Kx = Mg$.

   (c) (i) $T = (1/2)\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{q}},\quad V = (1/2)\mathbf{q}^{\mathrm{T}}\mathbf{K}\mathbf{q}$.

   (ii) $F = (1/2)\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{C}\dot{\mathbf{q}}$, (iii) not possible.