ENGINEERING TRIPOS PART IIA

Thursday 6 May 2010 9 to 10.30

Module 3C6

VIBRATIONS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
Data Sheet: 3C5 Dynamics and 3C6 Vibration (6 pages)

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Final Version

1 An H-shaped antenna and its mounting are represented schematically in Fig. 1. The four elements AC, AD, BE, BF are identical flexible beams with mass per unit length m and bending stiffness EI. The member AB is a massless rigid link which may be assumed to clamp the flexible elements at their attachment points A and B without allowing any rotation. The antenna can vibrate laterally in the plane of the diagram, so that the flexible elements execute small-amplitude bending motion. The supporting structure allows the link AB to move laterally, restrained by a spring.

(a) For the case in which the restraining spring is *infinitely stiff*, explain why all natural frequencies can be found by considering a single flexible element as a cantilever. Hence show that all the natural frequencies are determined by the roots of the equation

$$\cos\alpha L \cosh\alpha L = -1$$

where L is the length of the cantilever and α should be defined. [30%]

(b) Explain why the natural frequencies of the rigidly-fixed antenna in part (a) occur in groups of four. Sketch the first four modes in a form which takes account of the two-fold symmetry of the antenna. [20%]

(c) The infinitely stiff spring is now removed, so that the element AB is completely free to move in the horizontal direction.

(i) Explain why only mode of the group of four modes in part (b) has a different natural frequency from that given by part (a). Sketch this mode. [15%]

(ii) For the natural frequency that changes, explain how the new mode can be found by considering a single flexible element which has no rotation and zero shear force at the base (A or B). [10%]

(iii) Hence show that the new natural frequencies can be determined from the roots of the equation

$$\tan\alpha L = -\tanh\alpha L$$

where α and L are as in part (b) [25%]

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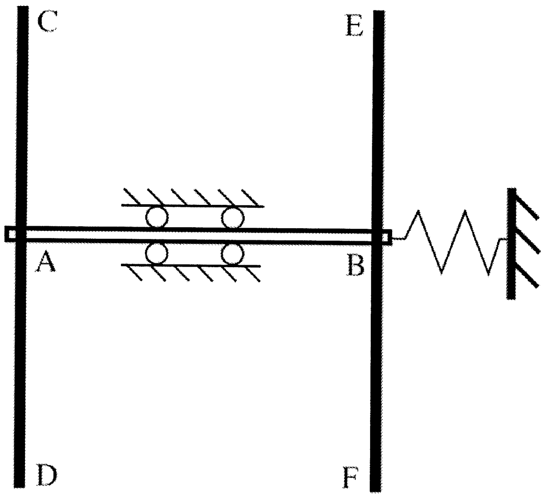

Fig. 1

2 An elastic column with cross-sectional area A, density ρ and Young's modulus E can undergo small axial vibrations, with displacement $u(x,t)$. The column is rigidly fixed at its base at $x = 0$, and is free at its top end at $x = L$.

(a) Write down suitable boundary conditions for the two ends of the column, and hence determine the mode shapes and natural frequencies of the column. Sketch the first three mode shapes. [40%]

(b) The column is held in tension by a force F applied by a cable attached to the top. Write down an expression for the static deformation of the column caused by this force. At time $t = 0$ the cable breaks. Explain how the step function response can be used to analyse this situation and hence find an expression for the subsequent transient vibration at a general point x. [50%]

(c) Modify the expression found in part (b) to incorporate the effect of small damping, and explain what happens as $t \to \infty$. [10%]

Hint: You may assume the Fourier series expansion:

$$x = \sum_{n=1}^{\infty} \frac{2L(-1)^{n+1}}{(n-1/2)^2 \pi^2} \sin \frac{(n-1/2)\pi x}{L} \qquad 0 \le x \le L$$

3 Figure 2 shows four rigid rods, each of mass m and length $2a$, freely pivoted at one end and able to rotate in a horizontal plane with small rotations $[\theta_1 \quad \theta_2 \quad \theta_3 \quad \theta_4]^{\mathrm{T}}$. The rods are joined by three springs of stiffness k as shown.

(a) Write expressions for the potential and kinetic energies of the system. [20%]

(b) Without detailed calculation, sketch the mode shapes of the system in order of increasing frequency. Explain salient features. [25%]

(c) Two of the mode shapes are symmetric. Write down the eigenvector of one of these modes by inspection and explain how orthogonality considerations can be used to deduce the other. Hence find both natural frequencies. [25%]

(d) The anti-symmetric modes have the form $[1 \quad \alpha \quad -\alpha \quad -1]^{\mathrm{T}}$ where α is a constant. Estimate values of α and the corresponding natural frequencies by finding the stationary points in Rayleigh's quotient. Comment on the accuracy of the natural frequencies and mode shapes found by this analysis. [30%]

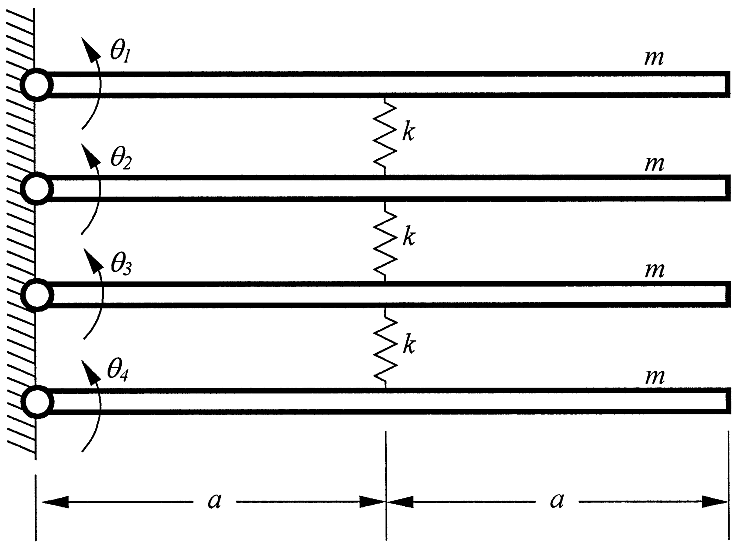

Fig. 2

4 Figure 3 shows a 'pitch-plane' model of a vehicle, subject to small vibrations. The wheels are represented by two masses m, which are constrained to move vertically with displacements x_1 and x_2. The body is symmetric, with mass $4m$ and pitch moment of inertia $I = ma^2$. It can move vertically with displacement y and pitch with angle θ. The masses are connected by four linear springs, each of stiffness k, representing the tyres and suspension springs. The distance between the two axles is $2a$.

(a) Assuming small motions, write down expressions for the kinetic and potential energies T and V in terms of the coordinates x_1, x_2, y and θ. Hence write down the mass matrix and show that the stiffness matrix can be written:

$$[K] = k\begin{bmatrix} 2 & 0 & -1 & a \\ 0 & 2 & -1 & -a \\ -1 & -1 & 2 & 0 \\ a & -a & 0 & 2a^2 \end{bmatrix}$$

where the vector of generalised coordinates is $[x_1 \quad x_2 \quad y \quad \theta]^{\mathrm{T}}$. [30%]

(b) The system has two natural modes with eigenvectors of the form $[1 \quad 1 \quad \alpha \quad 0]^{\mathrm{T}}$ and two natural modes with eigenvectors of the form $[1 \quad -1 \quad 0 \quad \beta]^{\mathrm{T}}$. Calculate the values of α and β, and determine the four natural frequencies. Sketch the four corresponding mode shapes. [40%]

(c) Sketch log amplitude plots for the transfer functions describing the displacements of coordinates x_1 and y when a sinusoidal force f is applied to the left wheel as shown in the figure. [30%]

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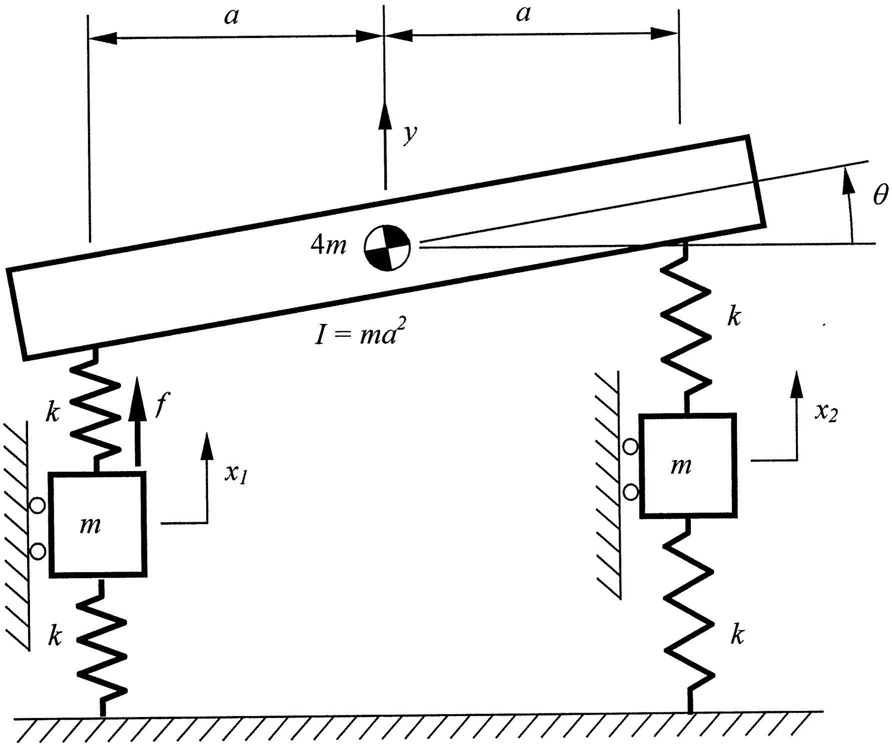

Fig. 3

END OF PAPER

ENGINEERING TRIPOS PART IIA

Module 3C6 Examination, 2010

Answers

2 (a) u(0,t)=0, u'(L,t)=0

(b) $u(x,t) = \dfrac{2LF}{EA\pi^2} \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{(n-1/2)^2} \sin\left(\dfrac{(n-1/2)\pi x}{L}\right) \cos\left(\dfrac{(n-1/2)\pi ct}{L}\right).$

(c) $u(x,t) = \dfrac{2LF}{EA\pi^2} \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{(n-1/2)^2} \sin\left(\dfrac{(n-1/2)\pi x}{L}\right) \cos\left(\dfrac{(n-1/2)\pi ct}{L}\right) e^{-\zeta_n \omega_n t}$

3 (a) $T = \dfrac{2}{3} ma^2 \left(\dot\theta_1^2 + \dot\theta_2^2 + \dot\theta_3^2 + \dot\theta_4^2\right),$

$V = \dfrac{1}{2} ka^2 \left[(\theta_1 - \theta_2)^2 + (\theta_2 - \theta_3)^2 + (\theta_3 - \theta_4)^2\right]$

(c) $[1\ 1\ 1\ 1]^{\mathrm{T}}, \quad \omega_1 = 0, \quad \omega_3 = \sqrt{\dfrac{3k}{2m}}$

(d) $\alpha = -1 \pm \sqrt{2}, \quad \omega_2 = 0.663\sqrt{\dfrac{k}{m}}, \quad \omega_4 = 1.60\sqrt{\dfrac{k}{m}}$, all frequencies are exact.

4 (a) $T = \dfrac{1}{2} m\dot x_1^2 + \dfrac{1}{2} m\dot x_2^2 + 2m\dot y^2 + \dfrac{1}{2} ma^2\dot\theta^2, \quad [M] = m\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & a^2 \end{bmatrix}.$

$V = \dfrac{1}{2} k\left[x_1^2 + x_2^2 + (y - a\theta - x_1)^2 + (y + a\theta - x_2)^2\right]$

(b) $\alpha = \dfrac{3}{4} \pm \dfrac{\sqrt{17}}{4}, \omega^2 = 0.219\dfrac{k}{m}, 2.281\dfrac{k}{m}; \quad \beta = \pm\dfrac{\sqrt{2}}{a}, \omega^2 = \left(2 \pm \sqrt{2}\right)\dfrac{k}{m}$