ENGINEERING TRIPOS PART IIA

Monday 3 May 2010 2.30 to 4.00

Module 3C7

MECHANICS OF SOLIDS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: 3C7 formulae sheet (2 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Final version

1 It has been suggested that stresses derived from the Airy Stress Function

$$\phi = -\frac{w}{4}x^2 - \frac{3w}{4d}x^2y + \frac{w}{20d^3}\left(2d^2y^3 - 5l^2y^3 + 20x^2y^3 - 4y^5\right)$$

may be suitable to describe the stresses within the simply-supported beam shown in Fig. 1. The beam is of length l between supports, and depth d, and carries a uniform pressure w on its top face.

(a) Calculate the stresses within the beam defined by ϕ, and show that they satisfy the boundary conditions on the top and bottom faces. [40%]

(b) Show that the stresses satisfy boundary conditions for simple supports at $x = \pm l/2$. Comment on the suitability of this solution for the particular supports shown in Fig. 1. [40%]

(c) How would you modify ϕ if the beam was also carrying a uniform tension along its length? [20%]

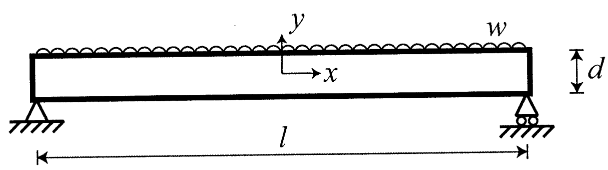

Fig. 1

2 (a) Derive the differential equilibrium equation relating the hoop stresses $\sigma_{\theta\theta}$ and the radial stresses σ_{rr} in a disk spinning at rotational speed ω. [20%]

(b) A flywheel is to be made from a lead alloy with density $\rho = 10 \times 10^3\,\mathrm{kg\,m^{-3}}$, Poisson's ratio $\nu = 0.3$, and uniaxial yield stress $Y = 12 \times 10^6\,\mathrm{N\,m^{-2}}$. It is assumed that the material will yield according to the Tresca yield criterion. Two designs are considered: each has a thickness of 10 mm and a diameter of 200 mm, but one is a continuous disk, while the other contains a central hole of diameter 20 mm.

(i) For each disk, find the rotational speed at which yield will first occur. [60%]

(ii) By considering the central hole as a stress concentrator, comment on the relative size of the stresses towards the centre of the two disks at a given rotational speed. [20%]

3 (a) Determine the stress state in polar coordinates (r,θ) from the Airy stress function

$$\phi = Cr\theta\sin\theta.$$ [20%]

(b) Show that the stresses found in (a) give the elastic solution for a half-space loaded by a line-load P, as shown in Fig. 2(a), and find the magnitude of the constant C. [30%]

(c) Show that the stress components σ_{rr} and $\sigma_{\theta\theta}$ are constant around a circle of diameter D centered at a distance $D/2$ below the point of loading. [20%]

(d) By superposition, find the stress component σ_{xx} at the centre of a cylinder of diameter D loaded by compressive loads at opposite ends of a diameter, as shown in Fig. 2(b) [30%]

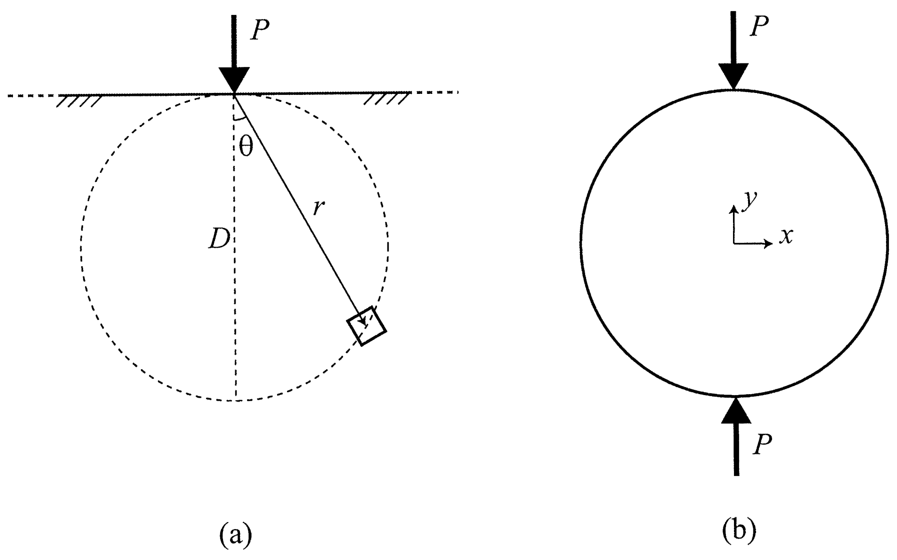

Fig. 2

4 In a plane strain backward extrusion process a billet of half-width $3a$ is compressed by a smooth, rigid punch of half-width $2a$ driven at speed v as shown in Fig. 3. The material of the billet can be considered to be rigid-perfectly plastic with a flow stress in shear of magnitude k. All the interfaces between the tooling and the deforming material are well lubricated. Velocity discontinuities for a possible Upper Bound analysis are indicated.

(a) Show that a reasonable estimate of the required punch pressure p in terms of k and the value of the ratio x/a is given by the minimum of the expression

$$2k\left(z+\frac{2}{z}\right), \quad \text{where} \quad z=\frac{x}{a}.$$ [40%]

(b) Explain why the equation in part (a) would not be appropriate for those situations in which the plastic zone extends over the distance y to the back face of the die. Derive an alternative equation for this case. [40%]

(c) At what position of the punch does your analysis in part (b) begin to provide a more attractive estimate of the extrusion pressure than the original estimate given in part (a)? [20%]

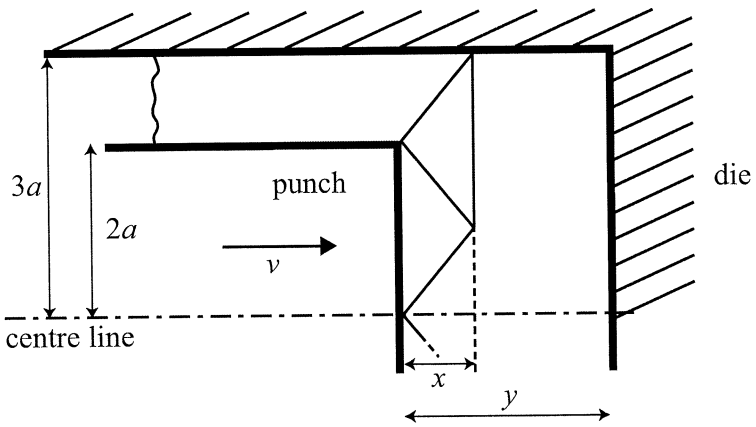

Fig. 3

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Final version

Module 3C7: Mechanics of Solids
ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

	Discs and tubes	Spheres
Equilibrium	$\sigma_{\theta\theta} = \dfrac{d(r\sigma_{rr})}{dr} + \rho\omega^2 r^2$	$\sigma_{\theta\theta} = \dfrac{1}{2r}\dfrac{d(r^2\sigma_{rr})}{dr}$
Lamé's equations (in elasticity)	$\sigma_{rr} = A - \dfrac{B}{r^2} - \dfrac{3+\nu}{8}\rho\omega^2 r^2 - \dfrac{E\alpha}{r^2}\int_c^r rT\,dr$	$\sigma_{rr} = A - \dfrac{B}{r^3}$
	$\sigma_{\theta\theta} = A + \dfrac{B}{r^2} - \dfrac{1+3\nu}{8}\rho\omega^2 r^2 + \dfrac{E\alpha}{r^2}\int_c^r rT\,dr - E\alpha T$	$\sigma_{\theta\theta} = A + \dfrac{B}{2r^3}$

2. Plane stress and plane strain

	Cartesian coordinates	Polar coordinates
Strains	$\varepsilon_{xx} = \dfrac{\partial u}{\partial x}$	$\varepsilon_{rr} = \dfrac{\partial u}{\partial r}$
	$\varepsilon_{yy} = \dfrac{\partial v}{\partial y}$	$\varepsilon_{\theta\theta} = \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$
	$\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$	$\gamma_{r\theta} = \dfrac{\partial v}{\partial r} + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r}$
Compatibility	$\dfrac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \dfrac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \dfrac{\partial^2 \varepsilon_{yy}}{\partial x^2}$	$\dfrac{\partial}{\partial r}\left\{r\dfrac{\partial \gamma_{r\theta}}{\partial \theta}\right\} = \dfrac{\partial}{\partial r}\left\{r^2\dfrac{\partial \varepsilon_{\theta\theta}}{\partial r}\right\} - r\dfrac{\partial \varepsilon_{rr}}{\partial r} + \dfrac{\partial^2 \varepsilon_{rr}}{\partial \theta^2}$
or (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}(\sigma_{xx} + \sigma_{yy}) = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}\right\}(\sigma_{rr} + \sigma_{\theta\theta}) = 0$
Equilibrium	$\dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} = 0$	$\dfrac{\partial}{\partial r}(r\sigma_{rr}) + \dfrac{\partial \sigma_{r\theta}}{\partial \theta} - \sigma_{\theta\theta} = 0$
	$\dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \sigma_{xy}}{\partial x} = 0$	$\dfrac{\partial \sigma_{\theta\theta}}{\partial \theta} + \dfrac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$
$\nabla^4\phi = 0$ (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}\left\{\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2}\right\} = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}\right\} \times \left\{\dfrac{\partial^2 \phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2}\right\} = 0$
Airy Stress Function	$\sigma_{xx} = \dfrac{\partial^2 \phi}{\partial y^2}$	$\sigma_{rr} = \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2}$
	$\sigma_{yy} = \dfrac{\partial^2 \phi}{\partial x^2}$	$\sigma_{\theta\theta} = \dfrac{\partial^2 \phi}{\partial r^2}$
	$\sigma_{xy} = -\dfrac{\partial^2 \phi}{\partial x \partial y}$	$\sigma_{r\theta} = -\dfrac{\partial}{\partial r}\left\{\dfrac{1}{r}\dfrac{\partial \phi}{d\theta}\right\}$

3. Torsion of prismatic bars

Prandtl stress function: $\sigma_{zx}\ (= \tau_x) = \dfrac{d\psi}{dy}$, $\quad \sigma_{zy}\ (= \tau_y) = -\dfrac{d\psi}{dx}$

Equilibrium: $T = 2\int_A \psi \, dA$

Governing equation for elastic torsion: $\nabla^2\psi = -2G\beta$ where β is the angle of twist per unit length.

4. Total potential energy of a body

$$\Pi = U - W$$

where $U = \frac{1}{2}\int_V \underset{\sim}{\varepsilon}^{\mathrm{T}}\,[D]\,\underset{\sim}{\varepsilon}\,\mathrm{d}V$, $\mathrm{W} = \underset{\sim}{P}^{\mathrm{T}}\,\underset{\sim}{u}$ and $[D]$ is the elastic stiffness matrix.

5. Principal stresses and stress invariants

Values of the principal stresses, σ_P, can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx}-\sigma_P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy}-\sigma_P & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\sigma_P \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_P.

Expanding: $\sigma_P{}^3 - I_1\,\sigma_P{}^2 + I_2\sigma_P - I_3 = 0$ where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix} .$$

6. Equivalent stress and strain

Equivalent stress $\bar{\sigma} = \sqrt{\frac{1}{2}}\,\{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\}^{1/2}$

Equivalent strain increment $\mathrm{d}\bar{\varepsilon} = \sqrt{\frac{2}{3}}\,\{\mathrm{d}\varepsilon_1{}^2 + \mathrm{d}\varepsilon_2{}^2 + \mathrm{d}\varepsilon_3{}^2\}^{1/2}$

7. Yield criteria and flow rules

Tresca

Material yields when maximum value of $|\sigma_1-\sigma_2|$, $|\sigma_2-\sigma_3|$ or $|\sigma_3-\sigma_1| = Y = 2k$, and then,

if σ_3 is the intermediate stress, $\mathrm{d}\varepsilon_1 : \mathrm{d}\varepsilon_2 : \mathrm{d}\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

von Mises

Material yields when, $(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{\mathrm{d}\varepsilon_1}{\sigma'_1} = \frac{\mathrm{d}\varepsilon_2}{\sigma'_2} = \frac{\mathrm{d}\varepsilon_3}{\sigma'_3} = \frac{\mathrm{d}\varepsilon_1-\mathrm{d}\varepsilon_2}{\sigma_1-\sigma_2} = \frac{\mathrm{d}\varepsilon_2-\mathrm{d}\varepsilon_3}{\sigma_2-\sigma_3} = \frac{\mathrm{d}\varepsilon_3-\mathrm{d}\varepsilon_1}{\sigma_3-\sigma_1} = \lambda = \frac{3}{2}\frac{\mathrm{d}\bar{\varepsilon}}{\bar{\sigma}} .$$

3C7 2010: Answers

1.(a) $\sigma_{xx} = \dfrac{w}{10d^3}\left[6d^2y - 15l^2y + 60x^2y - 40y^3\right];$

$\sigma_{yy} = \dfrac{w}{2d^3}\left[-d^3 - 3yd^2 + 4y^3\right];$

$\sigma_{xy} = \dfrac{w}{2d^3}\left[-12xy^2 + 3xd^2\right]$

2.(b.i) $539\,\mathrm{rad\,s^{-1}}$; $381\,\mathrm{rad\,s^{-1}}$

3.(a) $\sigma_{rr} = \dfrac{2C}{r}\cos\theta;\ \sigma_{\theta\theta} = 0;\ \sigma_{r\theta} = 0$

(d) $\sigma_{xx} = \dfrac{2P}{\pi D}$

4.(b) One mechanism gives $p = 2k\left(\dfrac{y}{a} + \dfrac{a}{y}\right)$

(c) $z = \sqrt{2} + 1$