ENGINEERING TRIPOS PART IIA

Thursday 6 May 2010 2.30 to 4

Module 3D7

FINITE ELEMENT METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: 3D7 Data Sheet (3 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Final version

1 Consider on the interval running from $x = 0$ to $x = L$ the problem governed by the differential equation

$$\alpha \frac{du}{dx} - \frac{d}{dx}\left(\beta \frac{du}{dx}\right) = f$$

where α and β are positive constants, and with the boundary conditions

$$u = 0 \quad \text{at} \quad x = 0$$

$$\beta \frac{du}{dx} = h \quad \text{at} \quad x = L$$

(a) Show that a weak version of this problem involves finding u such that

$$\int_0^L \alpha v \frac{du}{dx}\, dx + \int_0^L \beta \frac{dv}{dx}\frac{du}{dx}\, dx = \int_0^L v f\, dx + vh|_{x=L} \qquad (1)$$

holds for almost all test functions v, and state the restrictions on v. [25%]

(b) For a linear element of length l, compute the element stiffness matrix $\boldsymbol{k_e}$ for the formulation in Equation (1). [50%]

(c) Formulate an alternative version of the weak form to that in Equation (1) which would still be suitable for finite element analysis. [25%]

2 Figure 1(a) shows a triangular domain made of two isotropic materials with heat conductivities $k_1 = 3$ and $k_2 = 6$. The heat flux on the edge AC is prescribed to be $\overline{q} = 10$ and the edge CB is insulated. A constant temperature $\overline{T} = 10$ is prescribed on edge AB. The domain is discretized with two three-noded triangular elements as shown in Fig. 1(b).

(a) Compute the global source vector $\boldsymbol{f}$. [25%]

(b) Compute the global conductance matrix $\boldsymbol{K}$. [40%]

(c) Compute the temperature at the four finite element nodes and determine the corresponding heat flux across the domain edge CB. Briefly comment on the computed energy flux across the domain edge CB and suggest how its accuracy could be improved. [35%]

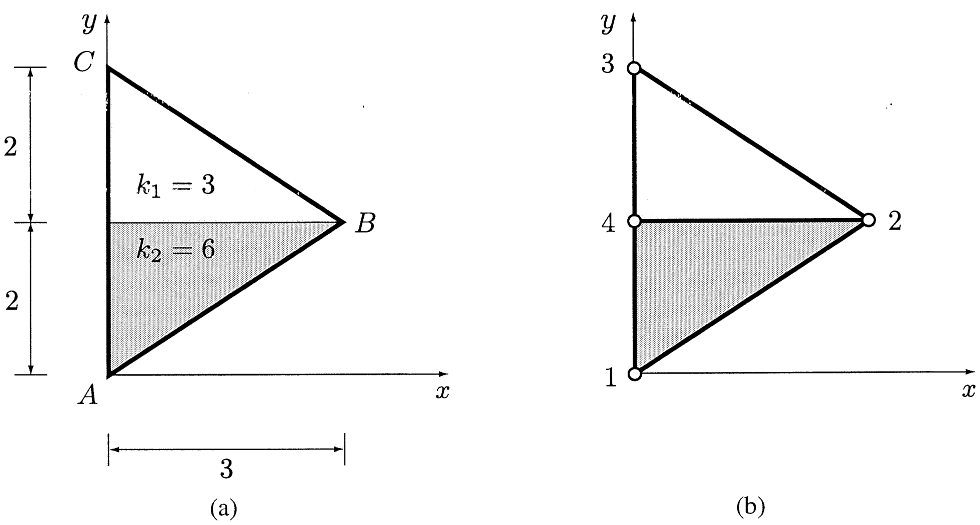


Fig. 1

3 (a) Figure 2(a) shows a four-noded elastic element with the following displacement vector

$$\boldsymbol{a}_e = \begin{bmatrix} u_{x1} & u_{y1} & u_{x2} & u_{y2} & u_{x3} & u_{y3} & u_{x4} & u_{y4} \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}^T$$

(i) Compute the strain components ε_{xx} and ε_{xy} along the top edge between nodes 3 and 4. [50%]

(ii) If the given displacements are due to an element force vector

$$\boldsymbol{f}_e = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}^T$$

comment if the strains found in (i) are physically meaningful. [20%]

(b) Consider the eight-noded hexahedral three-dimensional element shown in Fig. 2(b).

(i) Give an analytical expression for the shape function of node 1. [20%]

(ii) Suggest an integration rule for the numerical integration of the stiffness matrix using Gauss integration. Give reasons for your answer. [10%]

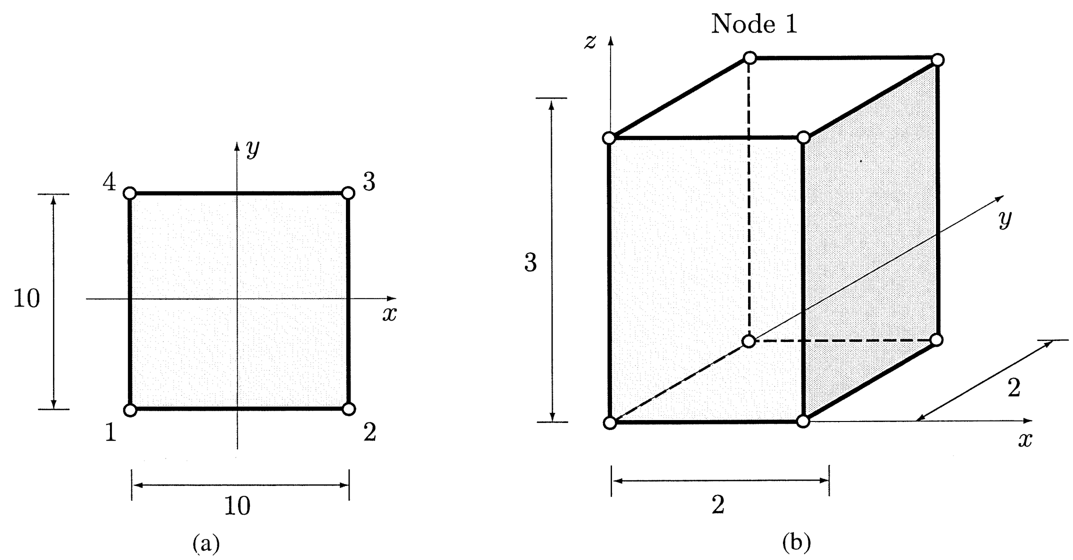


Fig. 2

4 (a) A particular semi-discrete finite element elastodynamics problem is in semi-discrete form written as

$$\boldsymbol{M}\ddot{\boldsymbol{a}} + \boldsymbol{K}\boldsymbol{a} = \boldsymbol{b} \tag{2}$$

Using the time stepping method

$$\frac{y_{n+1} - y_n}{\Delta t} = (1-\theta)\,\dot{y}_n + \theta\,\dot{y}_{n+1}$$

where θ is a constant and $0 \leq \theta \leq 1$:

(i) re-formulate equation (2) into a fully discrete problem such that it can be solved; [60%]

(ii) under what conditions can solving a system of equations be avoided for this problem? [10%]

(b) For a system of equations of the form $\boldsymbol{M}\dot{\boldsymbol{a}} + \boldsymbol{K}\boldsymbol{a} = \boldsymbol{b}$, the critical time step for a conditionally-stable time-stepping scheme is typically proportional to the square of a measure of the element size. For meshes with uniform element size, if the number of elements is doubled, estimate the change in the critical time step for a one-dimensional problem, a two-dimensional problem and a three-dimensional problem. [20%]

(c) Explain why the modal analysis approach is not applicable to non-linear time-dependent problems? [10%]

END OF PAPER

3D7 DATA SHEET

Element relationships

Elasticity

Displacement	$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{a}_e$
Strain	$\boldsymbol{\varepsilon} = \boldsymbol{B}\boldsymbol{a}_e$
Stress (2D/3D)	$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B}\, dV$
Element force vector (body force only)	$\boldsymbol{f}_e = \int_{V_e} \boldsymbol{N}^T \boldsymbol{f}\, dV$

Heat conduction

Temperature	$T = \boldsymbol{N}\boldsymbol{a}_e$
Temperature gradient	$\nabla T = \boldsymbol{B}\boldsymbol{a}_e$
Heat flux	$\boldsymbol{q} = -\boldsymbol{D}\nabla T$
Element conductance matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B}\, dV$

Beam bending

Displacement	$v = \boldsymbol{N}\boldsymbol{a}_e$
Curvature	$\kappa = \boldsymbol{B}\boldsymbol{a}_e$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T EI \boldsymbol{B}\, dV$

Elasticity matrices

2D plane strain

$$\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

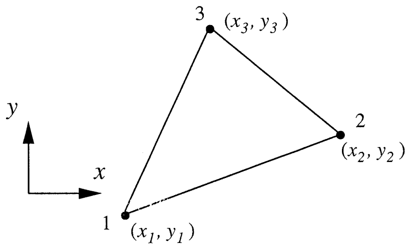
2D plane stress

$$\boldsymbol{D} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

Heat conductivity matrix (2D, isotropic)

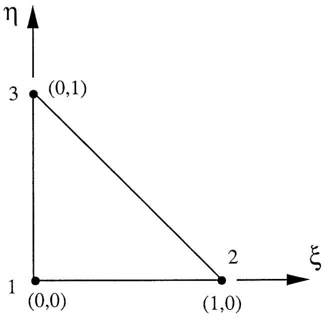
$$\boldsymbol{D} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

Shape functions

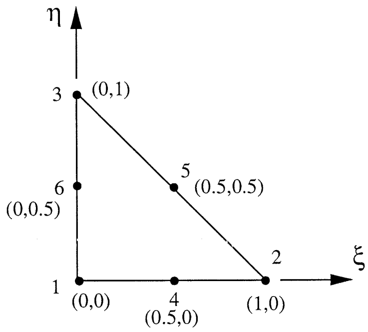


$$N_1 = ((x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y)/2A$$
$$N_2 = ((x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y)/2A$$
$$N_3 = ((x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y)/2A$$

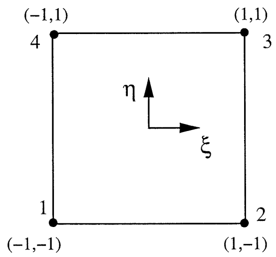
A = area of triangle



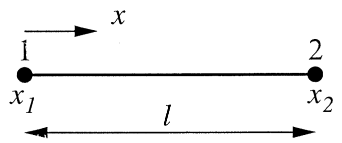
$$N_1 = 1 - \xi - \eta$$
$$N_2 = \xi$$
$$N_3 = \eta$$



$$N_1 = 2(1 - \xi - \eta)^2 - (1 - \xi - \eta)$$
$$N_2 = 2\xi^2 - \xi$$
$$N_3 = 2\eta^2 - \eta$$
$$N_4 = 4\xi(1 - \xi - \eta)$$
$$N_5 = 4\eta\xi$$
$$N_6 = 4\eta(1 - \xi - \eta)$$



$$N_1 = (1 - \xi)(1 - \eta)/4$$
$$N_2 = (1 + \xi)(1 - \eta)/4$$
$$N_3 = (1 + \xi)(1 + \eta)/4$$
$$N_4 = (1 - \xi)(1 + \eta)/4$$



Hermitian element

$$N_1 = \frac{-(x-x_2)^2(-l+2(x_1-x))}{l^3}$$

$$M_1 = \frac{(x-x_1)(x-x_2)^2}{l^2}$$

$$N_2 = \frac{(x-x_1)^2(l+2(x_2-x))}{l^3}$$

$$M_2 = \frac{(x-x_1)^2(x-x_2)}{l^2}$$

Gauss integration in one dimension on the domain $(-1,1)$ Using n Gauss integration points, a polynomial of degree $2n-1$ is integrated exactly.

number of points n	location ξ_i	weight w_i
1	0	2
2	$-\frac{1}{\sqrt{3}}$	1
	$\frac{1}{\sqrt{3}}$	1
3	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
	0	$\frac{8}{9}$
	$\sqrt{\frac{3}{5}}$	$\frac{5}{9}$

The list of numerical answers for 3D7

1b)

$$\boldsymbol{k}_e = \begin{bmatrix} -\frac{\alpha}{2} + \frac{\beta}{l} & \frac{\alpha}{2} - \frac{\beta}{l} \\ -\frac{\alpha}{2} - \frac{\beta}{l} & \frac{\alpha}{2} + \frac{\beta}{l} \end{bmatrix}$$

2a)

$$\boldsymbol{f}^T = \begin{bmatrix} -10 & 0 & -10 & -20 \end{bmatrix}$$

2b)

$$\boldsymbol{K} = \begin{bmatrix} \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \frac{9}{4} & -\frac{9}{4} \\ \text{---} & \text{---} & -\frac{9}{4} & \left(\frac{13}{4} + \frac{13}{4} \cdot 2\right) \end{bmatrix}$$

2c)

$$\begin{bmatrix} T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} -8.44 \\ -4 \end{bmatrix}$$

$$q_n = 1.7285$$

3a)

$$\epsilon_{xx} = \frac{1}{5}$$

$$\epsilon_{xx} = \frac{1}{50}$$

4b) 1D: Reduction factor 4
2D: Reduction factor 2
3D: Reduction factor 1.59