ENGINEERING TRIPOS PART IIA

Tuesday 27 April 2010 9 to 10.30

Module 3F1

SIGNALS AND SYSTEMS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Version: Final

1 Consider the feedback system of Fig. 1 where

$$P(z) = \frac{1}{z^2 + z + 0.9}.$$

(a) Draw the pole/zero diagram for the open–loop system $P(z)$. Is this system stable? [20%]

(b) Find the closed–loop transfer function from $R(z)$ to $Y(z)$ as a function of $K(z)$. [10%]

(c) For which values of $K(z) = k$ is the closed-loop system stable? [20%]

(d) Consider the closed–loop system and let the input r_n be an unit step. Find, as a function of $K(z) = k$, the steady–state value of y_n (i.e., the $\lim_{n\to\infty} y_n$) when this is finite, stating for which values of k the answer is valid. [20%]

(e) Let

$$K(z) = -\frac{1}{10}\frac{z-0.5}{z+0.5}.$$

Fig. 2 shows four Bode plots (A, B, C and D), but only one corresponds to $K(z)P(z)$. Choose the correct one, justifying your answer. [30%]

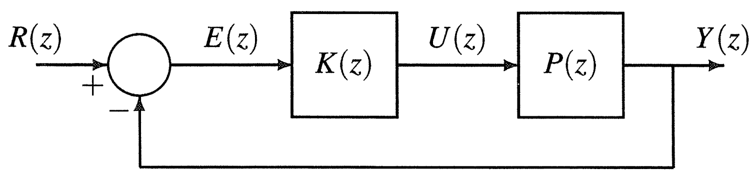

Fig. 1

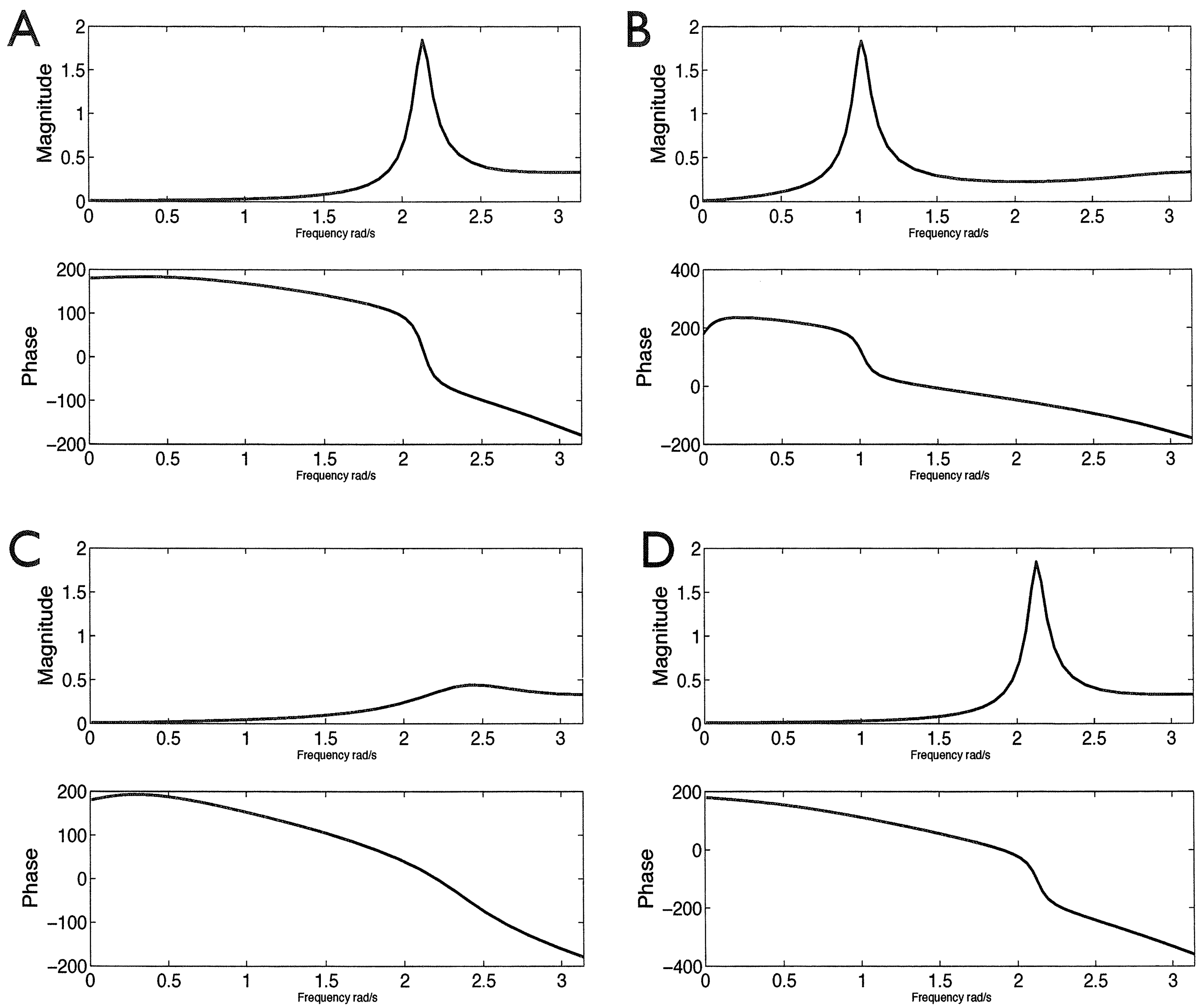

Fig. 2

2 (a) A continuous time system with transfer function $G(s)$ is connected to a digital-to-analogue converter (DAC) and an analogue-to-digital converter (ADC) as shown in Fig. 3. The DAC is a first order hold whose output consists of a linear extrapolation of the last two discrete inputs, and the sampling period is T seconds.

(i) Explain why the system relating $\{y(kT)\}$ to $\{u(kT)\}$ has a z-transfer function. [10%]

(ii) Show that the z-transfer function relating $\{y(kT)\}$ to $\{u(kT)\}$ is given by

$$H(z) = \frac{(z-1)^2}{Tz^2}\mathcal{Z}\left\{\left(\mathcal{L}^{-1}\left(G(s)V(s)\right)\right)_{t=kT}\right\}$$

where $V(s) = \frac{Ts+1}{s^2}$, $\mathcal{Z}$ stands for the z-transform and $\mathcal{L}^{-1}$ stands for the inverse Laplace transform. [40%]

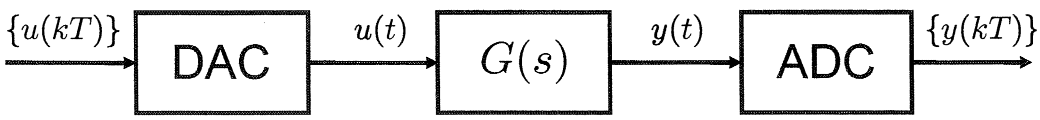

Fig. 3

(b) A random variable Y is generated from a random variable X using the monotonic increasing function $Y = g(X)$.

(i) Show that the probability density function (pdf) of Y, $f_Y(y)$, is related to the pdf of X, $f_X(x)$, by

$$f_Y(y) = \frac{f_X(x)}{g'(x)}$$

where $y = g(x)$ and $\frac{dy}{dx} = g'(x)$. [20%]

(ii) Let X have an exponential pdf

$$f_X(x) = \begin{cases} \frac{1}{d}\exp\left(\frac{-x}{d}\right) & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Determine $g(X)$ such that Y has a Rayleigh pdf

$$f_Y(y) = \begin{cases} \frac{y}{\sigma^2}\exp\left(\frac{-y^2}{2\sigma^2}\right) & \text{for } y \geq 0 \\ 0 & \text{for } y < 0 \end{cases}$$

[Hint: try $g(x) = ax^b$.] [30%]

3 (a) Define the terms *wide-sense stationary* (WSS) and *ergodic*, as applied to random processes, and state which of these is a pre-requisite for the other. [20%]

(b) Let $X(t)$ be an ergodic random process whose value at time t has a uniform probability density function (pdf) covering the range, $-A$ to A. The autocorrelation function (ACF) of X is triangular, of the form

$$r_{XX}(\tau) = \begin{cases} Q_X\left(1 - \frac{|\tau|}{T}\right) & \text{for } |\tau| \leq T \\ 0 & \text{for } |\tau| > T \end{cases}$$

where T is a constant. Determine the constant Q_X in terms of A. [20%]

(c) $X(t)$ is passed through a filter, with impulse response

$$h(t) = \delta(t) - \delta(t - T)$$

The output of the filter is $Y(t)$. Determine its ACF, $r_{YY}(\tau)$. [40%]

(d) Calculate $S_Y(\omega)$, the power spectral density of Y. [20%]

4 A first-order Markov message source generates states A, B and C with probabilities defined by the following conditional probability table:

$P(S_n|S_{n-1})$:

S_n \ S_{n-1}	A	B	C
A	0.1	0.2	0.7
B	0.7	0.1	0.2
C	0.2	0.7	0.1

where S_{n-1} and S_n represent the states of the source symbols at times $n-1$ and n.

(a) Explain what is meant by a first-order Markov source. [10%]

(b) Show that the equilibrium probabilities of this source at large n are given by $P_e = [1/3 \;\; 1/3 \;\; 1/3]$, and determine the mutual information in bits per symbol between S_n and S_{n-1}. [30%]

(c) Assuming three separate order-1 Huffman codes are used, one for each column of the above table, determine the codeword length for each entry in the table and hence determine the average number of bits per symbol required to encode the source in this way. [20%]

(d) The Huffman codes are now extended to order-2 so that, for each odd value of n, they encode S_{n+1} and S_n together, given S_{n-1}. Determine the average number of bits per symbol for this order-2 method and compare its coding efficiency with that of the above order-1 method. [30%]

(e) Explain briefly how arithmetic coding could be used to achieve even greater efficiency for this source. [10%]

END OF PAPER

Version: Final

Module 3F1, April 2009 – SIGNALS AND SYSTEMS – Answers

1 (a) Open-loop poles at $-0.5 \pm j0.8062$. Stable.
(b) $T(z) = K(z)/(z^2 + z + 0.9 + K(z))$.
(c) $-0.9 < k < 0.1$.
(d) For $-0.9 < k < 0.1$, $\lim_{n\to\infty} y_n = k/(2.9 + k)$.
(e) A.

2 (b) (ii) $g(x) = \sigma\sqrt{2/d}$.

3 (b) $Q_X = \frac{A^2}{3}$.
(d) $S_Y(\omega) = -4Q_X T \mathrm{sinc}^2(\omega T/2) - 4Q_X T \mathrm{sinc}^2(\omega T)$.

4 (b) $I(S_n; S_{n-1}) = 0.4282$ bits.
(c) $L = 1.3$ bits/sym.
(d) $L = 1.165$ bits/sym.