ENGINEERING TRIPOS PART IIA

Wednesday 5 May 2010 2.30 to 4

Module 3F2

SYSTEMS AND CONTROL

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Version: final

1 Figure 1 shows a bake-plate used for the heat-treatment of silicon wafers. It is divided into 3 segments. When a wafer is placed on the bake-plate, the temperature in contact with each segment is assumed to be uniform over the whole segment. The difference between the temperature of the i'th segment and the ambient air temperature is denoted by θ_i $(i = 1,2,3)$. Each segment has its own heater; power p_i is supplied to the heater of the ith segment, and can be varied.

Defining $x = [\theta_1, \theta_2, \theta_3]^T$ and $u = [p_1, p_2, p_3]^T$, the wafer temperatures evolve according to

$$\dot{x} = Ax + Bu$$

where

$$A = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -6 & 2 \\ 0 & 2 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(a) Show that this system is open-loop stable. [10%]

(b) What heater powers should be applied, in steady state, to hold the wafer temperature at 25°C above ambient in each segment? [10%]

(c) For a general square matrix A, show that if v is an eigenvector of A, then it is also an eigenvector of e^{At}. [30%]

(d) Verify that $v = [1,1,1]^T$ is an eigenvector of A for the wafer. Deduce that, if the wafer initially has a uniform temperature, then its temperature will remain uniform if $u = 0$, and give an expression for the evolution of this temperature with time. [20%]

(e) Suppose that state feedback of the form $u = -Kx$ is applied. Show that, if K is chosen such that $BK = cI$, where c is some constant, then the wafer temperature will remain uniform under closed-loop control, if the wafer temperature is initially uniform. [30%]

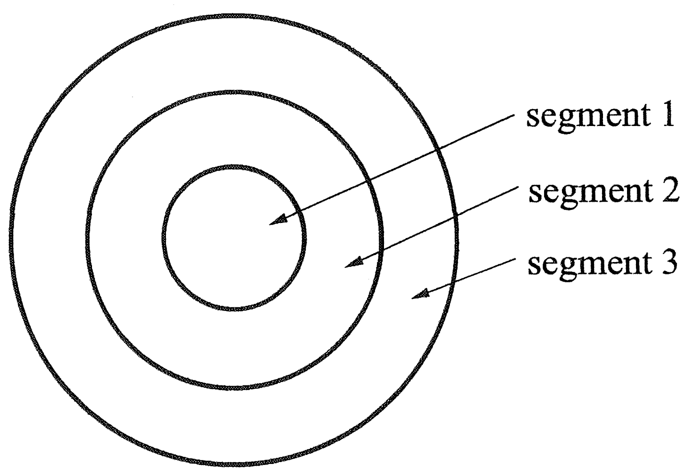

Fig. 1

2 Two chemical processes which are to be controlled have the transfer functions

$$G_1(s) = \frac{1}{(s+2)^4} \quad \text{and} \quad G_2(s) = \frac{s+4}{(s+2)^4}$$

respectively. Feedback controllers using proportional control only are to be used for both of them, as shown in Figure 2.

(a) Draw the root-locus diagram for $G_1(s)$, and show that it consists of straight-line segments only. [20%]

(b) Find the value k_1 of proportional gain required to obtain one pair of closed-loop poles with damping factor $1/\sqrt{2}$, with the transfer function $G_1(s)$. [20%]

(c) Sketch the root-locus diagram for $G_2(s)$. [30%]

(d) (i) If a stable feedback loop has return-ratio $L(s)$, explain why the steady-state error in response to a unit step demand signal is approximately $1/L(0)$ in most practical cases. [15%]

(ii) Show that if the gain k_1 is used with transfer function $G_1(s)$ then this approximation cannot be used, and find the steady-state error in this case. Comment briefly on the quality of the feedback design in this case. [15%]

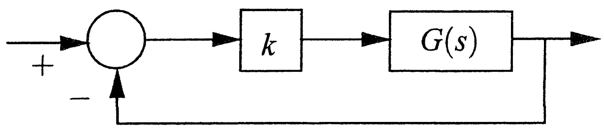

Fig. 2

3 The linearised equations for the pitch angle θ of an underwater vehicle are given by

$$\dot{x} = \begin{bmatrix} a_{11} & a_{12} \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} b_1 \\ 0 \end{bmatrix} \delta$$

where δ is the deflection of the stern plane, a_{11}, a_{12} and b_1 are non-zero constants, and the state vector is $x = [\dot{\theta}, \theta]^T$.

(a) Show that this system is controllable. [10%]

(b) The pitch angle is to be controlled to $\theta = 0$. Design a state-feedback system which will place the closed-loop poles at -1. [30%]

(c) In practice the equilibrium pitch angle with $\delta = 0$ may not be $\theta = 0$, because of a shift in the centre of gravity, imbalances of hydrodynamic forces, etc. Explain how integral action can be introduced into a state-feedback control scheme, and why it overcomes this problem. [35%]

(d) Design a state-feedback scheme for the underwater vehicle, incorporating integral action, which will place the closed-loop poles at -1, if $a_{11} = -2$, $a_{12} = 0.5$ and $b_1 = 1$. [25%]

4 A linear system is defined by the equations

$$\dot{x} = Ax + Bu, \qquad y = Cx \qquad (1)$$

where x, u, y are the state, input and output vectors, respectively.

(a) What is meant by the system being *observable*? How can one test whether a linear system is observable? [20%]

(b) If $u(t) = 0$ for all t, and the initial state is $x(0) = x_0$, show that

$$\int_0^\infty \|y(t)\|^2 dt = x_0^T W_o x_0$$

where W_o is the *observability Gramian* defined as

$$W_o = \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

(assuming that the system is asymptotically stable). [20%]

(c) The following equation arises in the study of some molecular structures:

$$\ddot{\theta} = -\varepsilon\theta - \dot{\theta} + \sqrt{2\varepsilon}u$$

where u is a disturbance signal. Only θ is measured. Put this into the standard form (1), and check that the resulting system is observable for all values of ε. [20%]

(d) (i) Show that the observability Gramian can be obtained as the solution to the matrix equation

$$A^T W_o + W_o A = -C^T C$$

(Hint: Consider $\frac{d}{dt}\left[e^{A^T t} C^T C e^{At}\right]$.) [30%]

(ii) Solve this equation to find W_o for the system you obtained in part (c). (Note that W_o is symmetric.) [10%]

END OF PAPER

Version: final

3F2 Systems and Control: 2010 Numerical answers

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12 May 2010

1. (b): $u = [10, 25, 25]^T$. (d): $x(t) = e^{-2t}\theta_0\nu$, where θ_0 is the initial temperature.
2. (b): $k_1 = 4$. (d)(ii): Steady-state error $= \frac{4}{5}r_0$ if r_0 is the amplitude of a step reference signal.
3. (b): $K = \left[\frac{a_{11}+2}{b_1}, \frac{a_{12}+1}{b_1}\right]$. (d): $K = [1, 3.5, -1]$.
4. (d)(ii): $W_o = \frac{1}{2\epsilon}\begin{bmatrix} 1+\epsilon & 1 \\ 1 & 1 \end{bmatrix}$.