ENGINEERING TRIPOS PART IIA

Thurs. 6 May, 2010 9.00 to 10.30

Module 3F3

SIGNAL AND PATTERN PROCESSING

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Final - 10 March 2010

1 (a) Define the power spectrum for a discrete time random process and give an interpretation for the power spectrum in terms of signal power. Detail any conditions that should be satisfied for a random process to have a power spectrum. [20%]

(b) If two wide-sense stationary random processes $\{A_n\}$ and $\{B_n\}$ are independent, show that the power spectrum of their product $X_n = A_n B_n$ can be expressed as a convolution:

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} S_A(e^{j\Omega})S_B(e^{j(\theta-\Omega)})d\Omega$$

where S_A and S_B are the power spectral densities for the two independent processes. [30%]

(c) A music synthesiser generates a sound by random amplitude modulation of a tone as follows:

$$X_n = (1 + A_n)\cos(n\omega_0 T + \phi)$$

where A_n is a zero mean, wide-sense stationary random process and ω_0 is a constant frequency.

Determine whether the process $\{X_n\}$ is wide-sense stationary when:

(i) $\phi = 0$

(ii) ϕ is a random variable, independent of $\{A_n\}$ and uniformly distributed between $-\pi$ and $+\pi$

[30%]

(d) Describe and sketch the power spectrum of the process $\{X_n\}$ with phase as in part (c)(ii) when

(i) $A_n = \cos(n\omega T/10 + \theta)$, and θ is a random variable, independent of ϕ, and uniformly distributed between $-\pi$ and $+\pi$.

(ii) $\{A_n\}$ is a second order autoregressive process having poles at $0.9\exp(\pm i\omega_0 T/10)$.

[20%]

[You may use the result that the DTFT of $\cos(\omega_0 nT)$ is $0.5\pi\delta(\omega - \omega_0)$ for values of ω between 0 and π/T.]

2 (a) The Discrete Fourier Transform (DFT) is typically implemented using a Fast Fourier Transform (FFT) algorithm. Assuming that the number of data points N is even, split the summation in the basic DFT equation into two parts: one for even n and one for odd n, and then show that the DFT values X_p and $X_{p+N/2}$ may be expressed as

$$X_p = A_p + W^p B_p \quad X_{p+N/2} = A_p - W^p B_p$$

Derive A_p, B_p and W in the above expression, and thus define the FFT "butterfly" structure. [30%]

(b) If N is a power of 2 the process above can be repeated several times resulting in a radix-2 FFT algorithm. Determine the number of such stages required, the number of "butterfly" computations per stage and the total number of complex multiplications if $N = 64$. [20%]

(c) A wide sense stationary discrete time random process $\{X_n\}$ has autocorrelation function

$$r_{xx}[l] = \begin{cases} 1, & l = 0 \\ 0.5, & l = \pm 1 \\ 0, & \text{otherwise} \end{cases}$$

The process is filtered through a first order noisy channel to give:

$$y_n = x_n - 0.2x_{n-1} + v_n$$

where v_n is white noise having variance 0.4.

(i) Determine the autocorrelation function of the noisy filtered process $\{Y_n\}$ and the cross-correlation function between $\{X_n\}$ and $\{Y_n\}$. [20%]

(ii) Determine the coefficients of a second order Wiener filter for extraction of $\{X_n\}$ from measurements $\{Y_n\}$. Compare its expected mean-squared error with the expected mean-squared error between $\{Y_n\}$ and $\{X_n\}$. [30%]

(TURN OVER

3 (a) Describe the direct form I structure for implementation of infinite impulse response (IIR) filters. Explain why implementing a digital filter in direct form I is satisfactory in Matlab where double precision floating-point is used but is not usually a good idea in fixed point implementation. What are other alternative realization structures of IIR filters and their potential advantages? Describe the direct form II structure and explain why it may be preferable to the direct form I implementation. [30%]

(b) The Goertzel algorithm is often used for detection and measurement of single sinusoidal tones since it computes a single Discrete Fourier Transform (DFT) component. The Goertzel algorithm is implemented as a second-order IIR filter with two real feedback coefficients and a single complex feedforward coefficient. The transfer function of the Goertzel filter is

$$H(z) = \frac{1+b_1z^{-1}}{1+a_1z^{-1}+z^{-2}}$$

where the filter coefficients for the mth bin of an N-point DFT are

$$b_1 = -\exp\left(-j\frac{2\pi}{N}m\right) \quad a_1 = -2\cos(2\pi m/N)$$

The filter is to be implemented in direct form II. Determine the coefficients of the filter for $m = 15$ and $N = 64$ and sketch the implementation. [30%]

(c) Show that the Goertzel filter in part (b) is equivalent to a first order complex all-pole filter having a single pole at $z = e^{+j\frac{2\pi}{N}m}$. Hence show that, assuming the input signal is zero prior to $n = 0$, the Goertzel algorithm delivers the expected DFT coefficient (up to a simple complex scale factor) after N data points have been passed through the filter. [40%]

4 Consider a binary classification problem with scalar real-valued observations x, and class labels $y \in \{0, 1\}$. Assume that $p(x|y=0)$ is a Gaussian distribution with mean 0 and variance 2, and $p(x|y=1)$ is a Gaussian distribution with mean 1 and variance 2. Furthermore, assume that $p(y=0) = p(y=1) = 1/2$.

(a) Compute the probability that given an observation $x = 2$, its corresponding class label is $y = 1$. [30%]

(b) Derive the general expression for $p(y=0|x)$ as a function of x, and discuss how this relates to logistic classification. [40%]

(c) Now assume that you fit a maximum likelihood Gaussian distribution $p(x|y=0)$ with mean μ_0 and variance σ_0^2 to the observed data with label $y = 0$, and similarly you fit a separate maximum likelihood Gaussian distribution $p(x|y=1)$ with mean μ_1 and variance σ_1^2 to the observed data with label $y = 1$.

Describe several ways in which the above procedure differs from maximum likelihood logistic classification, paying particular attention to the role of the variances and likelihood that is being optimised. [30%]

END OF PAPER