Q1 Q1.1

(a) $F(z) = \frac{m}{2\pi}\ln(z-ia) - \frac{m}{2\pi}\ln(z+ia) + Uz$

stagnation points are given by $dF/dz = 0$ where

$$\frac{dF}{dz} = \frac{m}{2\pi}\frac{1}{z-ia} - \frac{m}{2\pi}\frac{1}{z+ia} + U$$

$$= \frac{m}{2\pi}\frac{2ai}{z^2+a^2} + U = 0$$

$$\Rightarrow \quad z^2 = -a^2 - \frac{ima}{\pi U}$$

$$= a^2\left(1+\left(\frac{m}{\pi Ua}\right)^2\right)^{\frac{1}{2}} e^{i\theta}$$

where $\tan(\theta) = \frac{m}{\pi Ua}$ and $\pi \leq \theta \leq 3\pi/2$

Hence $z_1 = a\left(1+\left(\frac{m}{\pi Ua}\right)^2\right)^{\frac{1}{4}} e^{i\theta/2}$

$z_2 = a\left(1+\left(\frac{m}{\pi Ua}\right)^2\right)^{\frac{1}{4}} e^{i(\theta/2+\pi)}$

(b) when $m/\pi a U \ll 1 \Rightarrow$ Interaction between source and sink is weak. Near the source, flow pattern look like the flow composed of Uniform flow + source

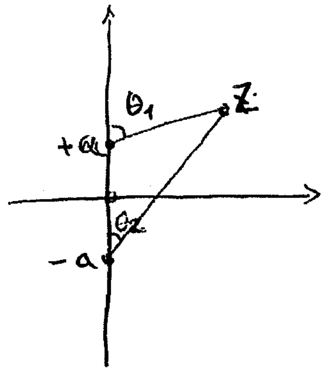
So pattern

Near the sink, flow pattern look like the flow composed of Uniform flow + sink

As the Interaction between source & sink is weak, the whole flow pattern is the ~~the~~ combination of the above two sketches.

Q1 (b) continue.

$$\psi(z) = \frac{m}{2\pi}(\theta_1 - \theta_2) + Uy$$



① Let $x(B), x(C) \to +\infty$

$\theta_1(B) \to 0, \quad \theta_1(C) \to 2\pi$

$\theta_2(B) \neq \theta_2(C) \to 0$

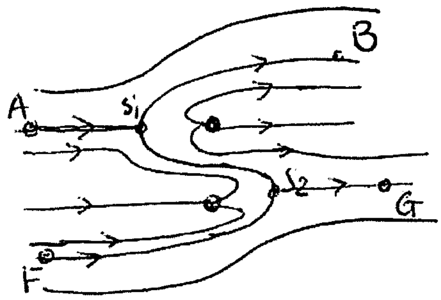
$$\Rightarrow \quad \psi(B) - \psi(C) = -m + U(y_B - y_C) = 0$$

② Let $x(E), x(F) \to -\infty$

$\theta_1(E), \theta_1(F), \theta_2(E), \theta_2(F) \to \pi$

$$\psi(E) - \psi(F) = U(y_E - y_F) = m.$$

(c) When AS_1C & ES_2G merge, streamline patterns

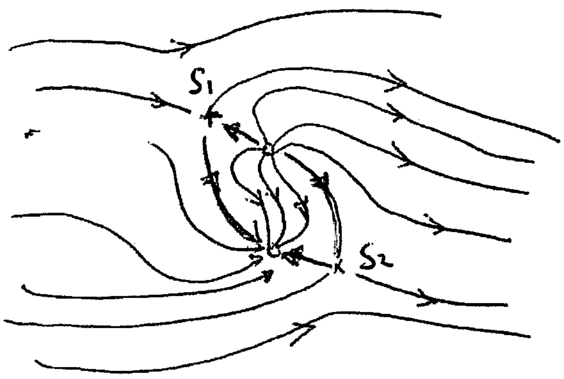


Let $x(A), x(F) \to -\infty$

$\psi(A) - \psi(F)$

$= U(y_A - y_F) = m.$

(d) when $m/\pi a U \gg 1$, streamline pattern.



For large $\frac{m}{\pi U a} \gg 1$, Streamlines

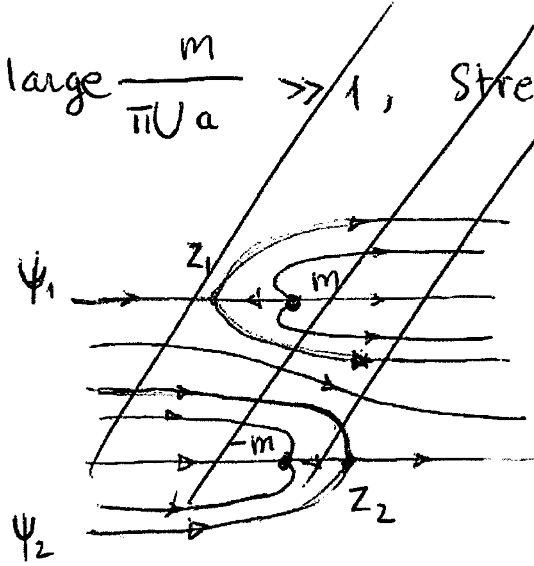


Fig.C

(e) The part of the upstream flows captured by the sink is shown in Fig.d. (the part between the two stagnation streamlines). If $\psi_1 - \psi_2 < m$ then the rest of the captured flow is made up by flow from the source.

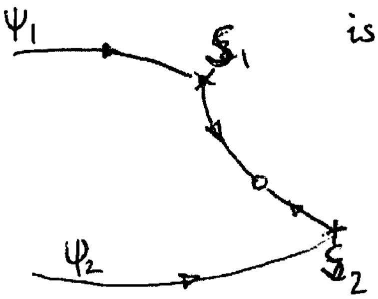
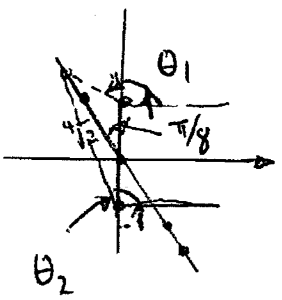


Fig.d

For the case $a = U = 1$, $m = \pi$, stream function

$$\psi = \frac{m}{2\pi}(\theta_1 - \theta_2) + Uy = \frac{1}{2}(\theta_1 - \theta_2) + y$$



Now $R = a\left(1 + \left(\frac{m}{\pi a U}\right)^2\right)^{\frac{1}{4}} = \sqrt[4]{2} = 1.189$

$\tan\theta = 1 \Rightarrow \theta = 5\pi/4 \Rightarrow \varphi = 5\pi/8$.

$\therefore\ S_1 = \left(\mp\sqrt[4]{2}\sin(\pi/8), \pm\sqrt[4]{2}\cos\pi/8\right) = (\mp 0.455, \pm 1.099)$

$\tan\theta_1 = -\frac{.099}{.455} \Rightarrow \theta_1 = 2.928$

$\tan\theta_2 = -\frac{2.099}{.455} \Rightarrow \theta_2 = 1.784$

$\therefore$ for S_1 $\psi_1 = \frac{1}{2}(2.928 - 1.784) + 1.189 = 1.676$

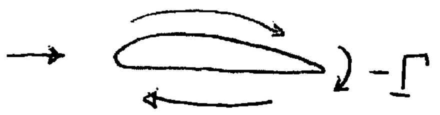
S_2 $\psi_2 = \frac{1}{2}(4.296 - 6.070) - 1.189 = -1.670$

$\psi_1 - \psi_2 = 3.341 > \pi \Rightarrow$ Nothing captured.

Q2 (a) Circulation around a curve $\Gamma = \oint \underline{u} \cdot d\underline{l}$

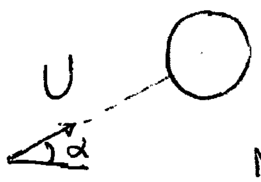
$(= \int \vec{\omega} . d\vec{s})$ where $\underline{\omega} = \nabla \times \underline{u}$

When a body in a mean flow has a non-zero circulation, it means the average flow speed is higher over one side than the other (i.e., mean pressure is lower on one side than the other) and a net lift force is exerted on the body.



It can be shown that the lift force $= \rho U(-\Gamma)$ and it is $\perp$ to the mean flow.

(b)



For large r, $\psi \sim U \sin(\theta - \alpha) r$ which is a Uniform flow at angle α to the x-axis.

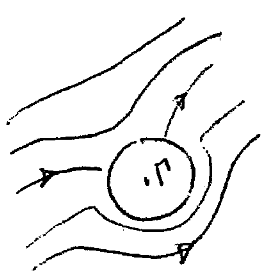
Now $\psi = U\left(r - \frac{a^2}{r}\right) \sin(\theta - \alpha) - \frac{\Gamma}{2\pi} \ln r$

because $\psi = 0 - \frac{\Gamma \ln a}{2\pi}$ when $r = a$, i.e. ψ is constant

i.e. the cylinder $r = a$ is a streamline.

Hence ψ can represent the flow around a cylinder.

(c) On the surface of the cylinder $u_r = 0$



$$\Rightarrow u = u_\theta = -\left.\frac{\partial \psi}{\partial r}\right|_{r=a} = \left.\left(U\left(1 + \frac{a^2}{r^2}\right)\sin(\theta - \alpha) - \frac{\Gamma}{2\pi}\frac{1}{r}\right)\right|_{r=a}$$

$$= 2U\sin(\theta - \alpha) - \frac{\Gamma}{2\pi a}$$

At a stagnation point $u = 0$

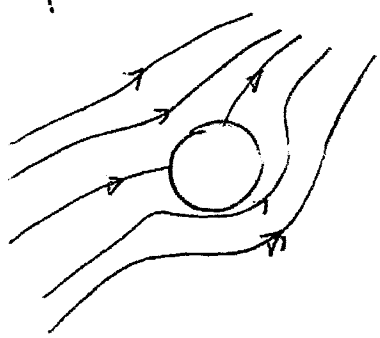
$$\Rightarrow 2U \sin(\theta - \alpha) = \frac{\Gamma}{2\pi a} \Rightarrow \sin(\theta - \alpha) = \frac{\Gamma}{4\pi a U}$$

Provide $\frac{\Gamma}{4\pi a U} < 1 \Rightarrow \begin{cases} \theta = \alpha + \sin^{-1}\left(\frac{\Gamma}{4\pi a U}\right) \\ \text{or } \theta = \alpha + \pi - \sin^{-1}\left(\frac{\Gamma}{4\pi a U}\right) \end{cases}$

(d) Lift force = force on equivalent vortex

$$= \rho U(-\Gamma) \quad \text{or} \quad |\text{lift}| = \rho U \Gamma$$

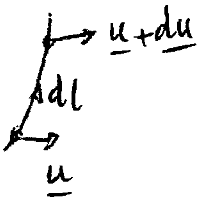
(~~e~~)



(e) In practice flow will separate (i.e. boundary layers will form and these will separate). This produces a great deal of drag. and very little lift.

If the cylinder is rotating, however, (the inviscid solution is unaltered by cylinder rotation) then something approximating the inviscid situation will be observed.

(a) Vortex lines move with fluid. Consider a vortex line Q3.1



stream (i.e. a line // vorticity). The two ends of the segment $\underline{dl}$ are moving at different velocities and so the vortex line is "stretched".

$u = u(x, y)$

$$\frac{d\,\underline{dl}}{dt} = \delta \underline{u} = \underline{\delta l} \cdot \nabla \underline{u}$$

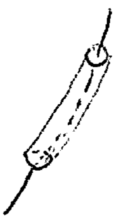
$|u| = |w| << 1$ neglected.

Since the total vortex strength remains constant through a material curve this results in the magnitude of vorticity increasing due to the stretch and a consequent inverse in the total rate of spin of the fluid.

$$\text{Kelvin} \Rightarrow \frac{d}{dt}\oint_C \underline{u}\,\underline{dl} = 0 \Rightarrow \Gamma = \text{constant} = \int_S \underline{w}\,\underline{dA}$$

So for a small material curve

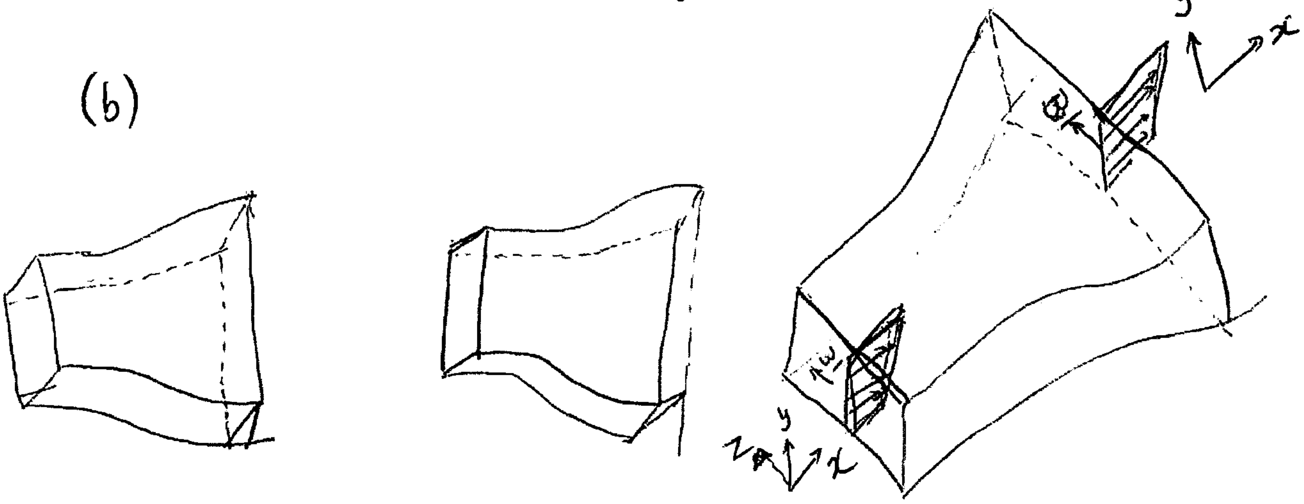
$\Gamma = \underline{w} \cdot \underline{dA}$ remains constant.



But conservation of mass $\Rightarrow$ if the flow is stretching the cylinder shown then $\underline{dA}$ is reducing $\Rightarrow$ $\underline{w}$ increases

In general $\dfrac{w}{l} = \dfrac{w_0}{l_0}$

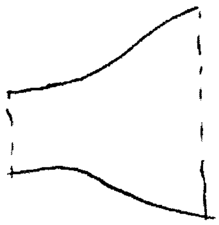
(b)



Q3.2.

(b) consider a vortex line shown. As it passes through the diffuser $\Rightarrow$ length doubles

$\Rightarrow$ vorticity doubles



At the inlet $|\text{vorticity}| = \dfrac{du}{dy} = \dfrac{U}{3h}$

At the exist $\dfrac{du}{dy} = \dfrac{2U}{3h}$

At exist then, $u = U_2 + \dfrac{2Uy}{3h}$

Conservation of mass

$$\Rightarrow \int_0^h u|_{inlet}\, a\, dy = \int_0^h u|_{exist}\, 2a\, dy$$

$$\Rightarrow Ua\int_0^h \left(1+\frac{y}{3h}\right)dy = 2aU_2h + 2a\int_0^h \frac{2Uy}{3h}\,dy$$

$$\Rightarrow Ua\left(h+\frac{h}{6}\right) = 2aU_2h + \frac{4Ua}{3h}\frac{h^2}{2}$$

$$\Rightarrow 2U_2 = \frac{7U}{6} - \frac{2U}{3} = \frac{U}{2} \Rightarrow U_2 = \frac{U}{4}.$$

Stream-lines @ inlet & outlet are // $\Rightarrow$ pressure uniform

$$\Rightarrow y=0 \quad \therefore \quad P_1 + \frac{\rho U^2}{2} = P_2 + \frac{\rho U^2}{2\times 16}$$

$$\Rightarrow P_2 - P_1 = \frac{15\rho U^2}{32}$$

(c) Assuming gradual changes $\Rightarrow$ no separation off side walls

if outlet area is $\lambda a \Rightarrow \dfrac{du}{dy}\Big|_{outlet} = \dfrac{\lambda U}{3h} \Rightarrow u = \dfrac{\lambda U y}{3h} + U_2$

Continuity $\Rightarrow \dfrac{7Uah}{6} = \lambda a U_2 h + \lambda a\displaystyle\int_0^h \frac{\lambda U}{3h}\, y\, dy = \lambda a U_2 h + \lambda^2\frac{Uah}{6}$

$$\Rightarrow U_2 = \frac{\frac{7Uah}{6} - \frac{\lambda^2 aUh}{6}}{\lambda ah}$$

U_2 = flow on floor $\to 0$ when $\lambda = \sqrt{7}$

Q 4.1

Question 4

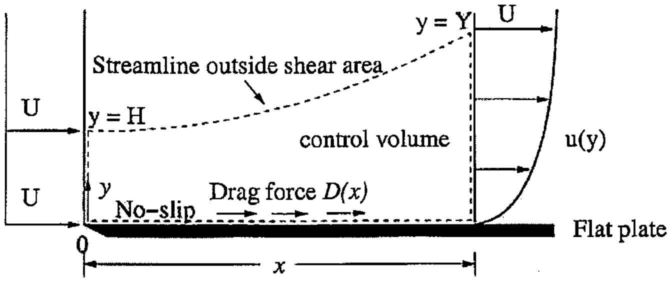


Figure 1: Definition of the control volume for analysis of flow past a flat plate.

(a) Assume the flow is steady, apply the conservation of mass to the control volume :

$$\int_0^Y \rho u dy - \int_0^H \rho U dy = 0$$

Assuming incompressible flow (constant density), this relation simplifies to

$$HU = \int_0^Y u dy = \int_0^Y (U + u - U) dy = YU + \int_0^Y (u - U) dy$$

Rearranging this and noting that $Y = H + \delta^*$, we can express the mass-f;ow relation in the following simple manner:

$$U(Y - H) = U\delta^* = \int_0^Y (U - u) dy$$

or

$$\delta^* = \int_0^{Y \to \infty} \left(1 - \frac{u}{U}\right) dy$$

This is the formal definition of the boundary-layer displacement thickness δ^* and holds true for any incompressible flow, whether laminar or turbulent, constant or variable pressure, constant or variable temperature. In other words, to define δ^* is simply to state conservation of mass in steady flow. Note that since the y variations are integrated away, δ^* is a function only of x. Its exact value depends upon the distribution $u(y)$.

(b) Apply the conservation of x-momentum to our control volume:

$$\sum F_x = -D = \int\int_{CS} u(\rho \mathbf{V} \cdot d\mathbf{A} = \int_0^Y u(\rho u dy) - \int_0^H U(\rho U dy)$$

or

$$\text{Drag} = D = U \int_0^H \rho U dy - \int_0^Y \rho u^2 dy$$

Again assuming constant ρ, and

$$\int_0^H \rho U dy = \int_0^Y \rho u dy,$$

we obtain, per unit depth,

$$\text{Drag} = \rho \int_0^Y u(U - u) dy$$

or

$$\frac{D}{\rho U^2} = \theta = \int_0^{Y\to\infty} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

This equation is the defining relation of the momentum thickness θ, which, like δ^*, is clearly a function of x only.

(c) If $\tau_w(x)$ is the local shear stress on the plate, the total drag per unit width on one side of a plate of length L is the integral of the wall shear forces:

$$D = \int_0^L \tau_w(x) dx$$

(d)

$$C_D = \frac{\int_0^L \tau_w dx}{\frac{1}{2}\rho U^2 L} = \frac{2\theta(L)}{L},$$

$$C_f = \frac{d}{dx}\left[xC_D(x)\right] = 2\frac{d\theta}{dx}$$

These formulas were derived by Kármán in his classic paper. They are valid for either laminar or turbulent flow. It is interesting that flat-plate friction and drag boil down to the determination of the momentum thickness $\theta(x)$.

Question 5

(a) $u(0)=0 \Rightarrow$ no-slip condition at the wall.

$u(\delta)=U \Rightarrow$ matching of the velocity profile ~~with~~ with external velocity.

$\frac{\partial u}{\partial y}(\delta)=0 \Rightarrow$ requirement for smooth transition at the outer edge of the boundary layer.

(b) For steady flow within the boundary layer

$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

$\frac{dU}{dx}=0$, combined with previous conditions $\Rightarrow \frac{\partial^2 u}{\partial y^2}=0$ at $y=0$ and $y=\delta$.

(c) $u(0)=0 \Rightarrow a_0=0$

$\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0}=0 \Rightarrow a_2=0$

The other 3 conditions $\Rightarrow a_1=2,\ a_3=-2,\ a_4=1$

Hence $\frac{u}{U} = 2\left(\frac{y}{\delta}\right)-2\left(\frac{y}{\delta}\right)^3+\left(\frac{y}{\delta}\right)^4$

(d) Momentum thickness

$$\theta = \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy = \int_0^\delta \left[2\left(\frac{y}{\delta}\right)-2\left(\frac{y}{\delta}\right)^3+\left(\frac{y}{\delta}\right)^4\right]\left[1-2\left(\frac{y}{\delta}\right)+2\left(\frac{y}{\delta}\right)^3-\left(\frac{y}{\delta}\right)^4\right]dy$$

$$\overset{\eta=y/\delta}{=} \delta\int_0^1 (2\eta-2\eta^3+\eta^4)(1-2\eta+2\eta^3-\eta^4)\,d\eta$$

$$= \delta\int_0^1 2\eta-4\eta^2-2\eta^3+9\eta^4-4\eta^5-4\eta^6+4\eta^7-\eta^8\,d\eta$$

$$= \delta\left(\eta^2-\frac{4}{3}\eta^3-\frac{\eta^4}{2}+\frac{9}{5}\eta^5-\frac{2}{3}\eta^6-\frac{4}{7}\eta^7+\frac{\eta^8}{2}-\frac{\eta^9}{9}\right)\Bigg|_0^1 = \frac{37}{315}\delta$$

Question 5

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)\Big|_{y=0} = \mu\frac{2U}{\delta}$$

$$C_f' = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{4\nu}{U\delta}$$

(e) Momentum integral equation:

$$\frac{d\theta}{dx} + \frac{(H+2)\theta}{U}\frac{dU}{dx} = \frac{C_f'}{2}$$

but $\frac{dU}{dx} = 0$

$$\Rightarrow \quad \frac{d\theta}{dx} = \frac{C_f'}{2}$$

(f) From (d), $\theta = \frac{37}{315}\delta$, $C_f' = \frac{4\nu}{U\delta}$

$$\frac{37}{315}\frac{d\delta}{dx} = \frac{2\nu}{U\delta}$$

$$\frac{d(\delta^2)}{dx} = 34.054\frac{\nu}{U}$$

$$\delta^2 = \overset{34.054}{\cancel{5.836}}\frac{\nu x}{U} + C$$ where C is a constant

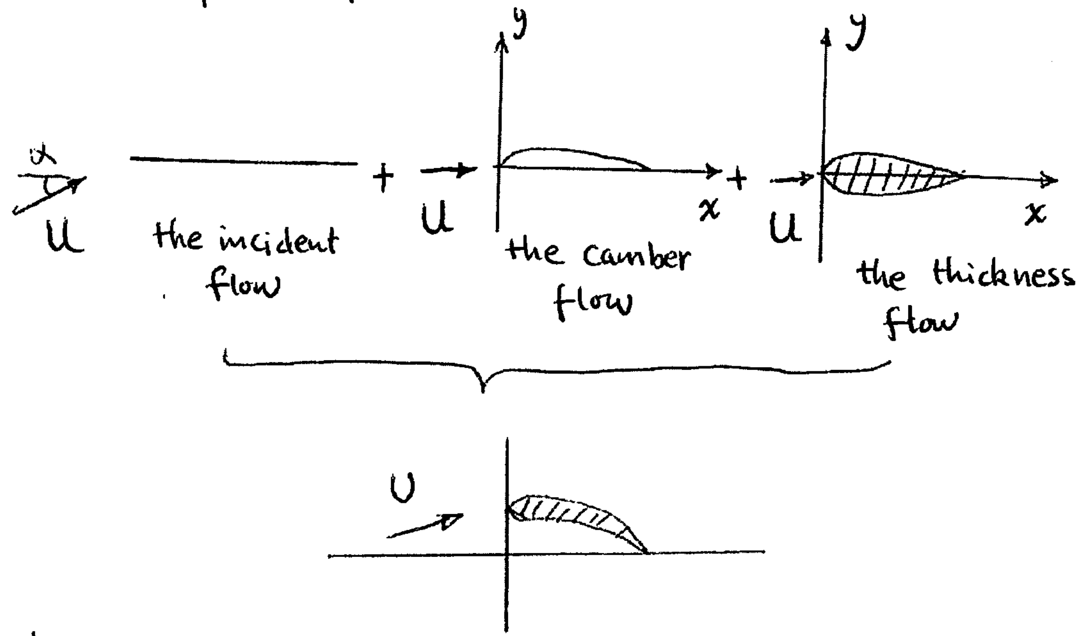
but $\delta(0) = 0 \Rightarrow C = 0$.

Finally, $$\delta = 5.836\sqrt{\frac{\nu x}{U}} = \frac{5.836}{\sqrt{\frac{Ux}{\nu}}}x$$

$$= \frac{5.836x}{\sqrt{Re_x}}$$

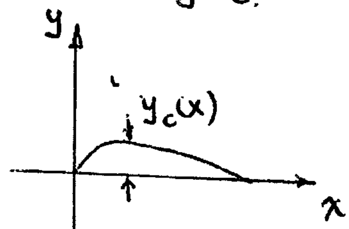
Question 6

(a) The basis of classical thin-airfoil theory is to superpose three component flows.



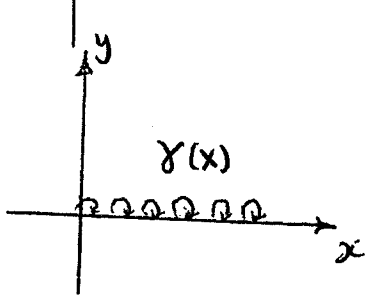
(b) The camber component is modelled by a chordwise distribution of vorticity $\gamma(x)$ per unit length and y_c, the camber line, is assumed small so that the vortex sheet is on $y=0$.

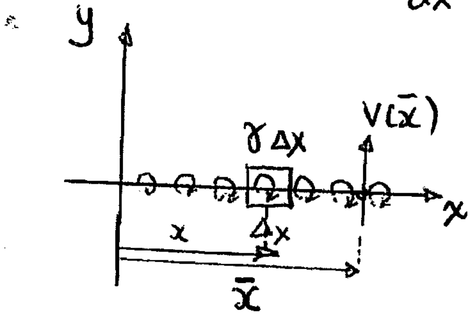
Hence $$v(\bar{x}) = \int_0^c \frac{\gamma(x)}{2\pi(\bar{x}-x)}\,dx$$

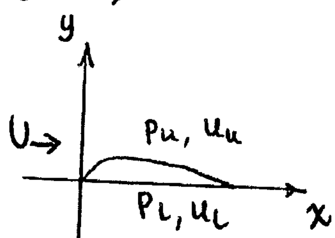


As boundary condition, $\dfrac{v}{U} = \dfrac{dy_c}{dx}$

Hence $$\frac{dy_c}{dx} = \frac{1}{2\pi U}\int_0^c \frac{\gamma(x)}{\bar{x}-x}\,dx$$







(c) The pressure of a point on the airfoil is

$$P = P_0 + \frac{\rho U^2}{2} - \frac{\rho}{2}\left((U+u)^2+v^2\right) = P_0 - \rho U u - \cancel{\frac{\rho}{2}(u^2+v^2)}$$

second order term

Hence the lift coefficient is $$C_L = \frac{1}{\frac{1}{2}\rho U^2 c}\int_0^c (P_l - P_u)\,dx = \frac{1}{\frac{1}{2}\rho U^2 c}\int_0^c \rho U (u_u - u_l)\,dx$$

but $\gamma \Delta x = \oint \vec{u}\cdot d\vec{l} = (u_l - u_u)\Delta x$ Que 6.2

$$\therefore \quad C_L = \frac{2}{Uc}\int_0^c \gamma(x)\,dx$$

C.f. Kutta–Joukowski $L = \rho U \Gamma = \rho U \int_0^c \gamma\,dx$

so $C_L = \frac{2}{Uc}\int_0^c \gamma(x)\,dx$.

(d) The form of γ distribution for a camber airfoil at incident α is (Data Sheet)

$$\gamma = -U\left\{(2\alpha + g_0)\frac{1-\cos\phi}{\sin\phi} + \sum_{n=1}^{\infty} g_n \sin(n\phi)\right\}$$

with $x = \frac{c}{2}(1+\cos\phi)$ and $dx = -\frac{c}{2}\sin\phi\,d\phi$

Hence $$C_L = \frac{2}{Uc}\int_0^{\pi} U\frac{c}{2}\left[(2\alpha+g_0)(1-\cos\phi) + \sum_{n=1}^{\infty} g_n \sin(n\phi)\sin\phi\right]d\phi$$

$$= (2\alpha + g_0)\underbrace{\int_0^{\pi}(1-\cos\phi)\,d\phi}_{\pi} + g_1\underbrace{\int_0^{\pi}\sin^2\phi\,d\phi}_{\pi/2}$$

since $\int_0^{\pi}\sin(n\phi)\sin\phi = 0$ for $n>1$

$$\therefore \quad C_L = \underbrace{2\pi\alpha}_{\text{flat plat}} + \pi\underbrace{\left(g_0 + \tfrac{1}{2}g_1\right)}_{\text{camber}}$$

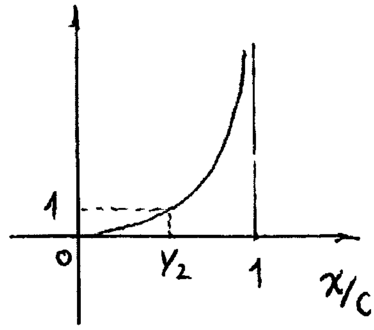
(e) Substituting for g_0 and g_1 from Data Sheet:

$$g_0 + \tfrac{1}{2}g_1 = \frac{1}{\pi}\int_0^{\pi}\left(-\frac{2dy_c}{dx}\right)(1+\cos\theta)\,d\theta$$

$$= \frac{2}{\pi c}\int_0^{c}\left(\frac{2dy_c}{dx}\right)\frac{1+\cos\theta}{\sin\theta}\,dx$$

$$= \frac{2}{\pi c}\int_0^{c}\underbrace{\left(\frac{2dy_c}{dx}\right)}_{\text{camber slop}}\underbrace{\sqrt{\frac{x/c}{1-x/c}}}_{\text{weighting}}\,dx$$

$\sqrt{\dfrac{x/c}{1-x/c}}$



$\therefore$ the influence of camber is greatest towards the rear of the airfoil (hence flaps...).

(f) The leading term is $\frac{1-\cos\phi}{\sin\phi}$, which is different from the other term Fourier $\sin(n\theta)$. The leading term is needed, because the experimental evidence shows a large suction peak at the airfoil's leading edge, which can be modeled by a function whose value is large at the leading edge and reduces to 0 at the trailing edge. $\frac{1-\cos\phi}{\sin\phi}$ satisfies this requirement. None of the Fourier terms does.

Question 7

(a) A basic open-loop wind-tunnel is shown in Fig. 2.

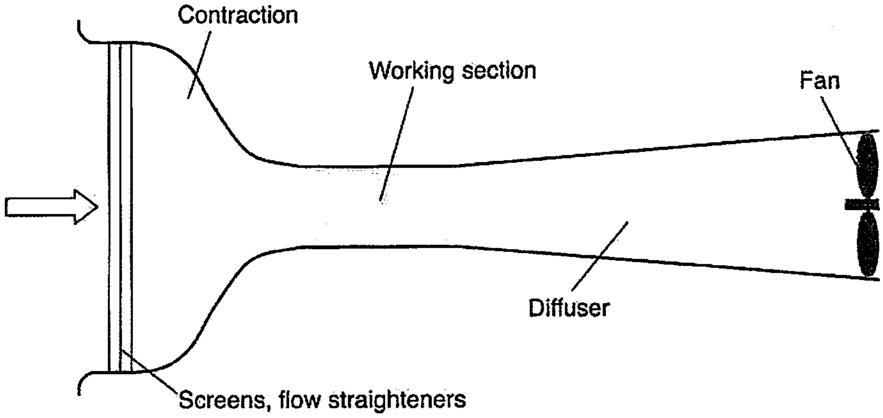
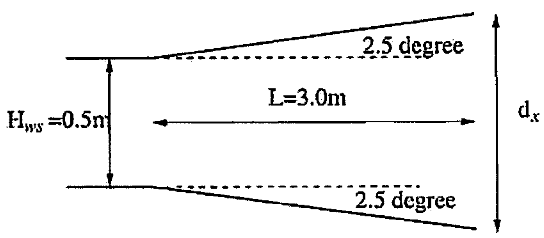


Figure 2: Basic open-loop wind-tunnel (from the lecture notes)

(b) Wind-tunnel power is equal to flow of kinetic energy at exit

$$\frac{1}{2}\rho U_x^3 A_x.$$

Assume $\rho = 1.225\ kg/m^3$. To find exit area, we assume an (max) included angle of 5^o in the diffuser. This gives an exit diameter



$$d_x = H_{ws} + L \times 2\tan(2.5^o) = 0.76m$$

$$A_x = \frac{\pi}{4} d_x^2 = 0.454\ m^2$$

From continuity:

$$A_{ws}U_{ws} = A_x U_x$$

$$U_x = \frac{A_{ws}}{A_x}U_{ws} = \frac{0.5^2}{\frac{\pi}{4}(0.76)^2}20\ m/s = 11.0\ m/s$$

Power

$$P = \frac{1}{2}\rho U_x^3 A_x = 370.0\ \text{W}$$

. Power factor

$$\lambda = \left(\frac{A_{ws}}{A_x}\right)^2 = 0.3$$

The main losses are likely be: fan losses, losses due to grid drag ahead of contraction and flow straightener, wall friction and model drag.

(c) The PIV system set-up in shown in Fig. 3. Needs windows in floor and on side. Light sheet erected by cylindrical lens. Light sheet enters from one side, e.g., floor. Camera observes light sheet from perpendicular direction.

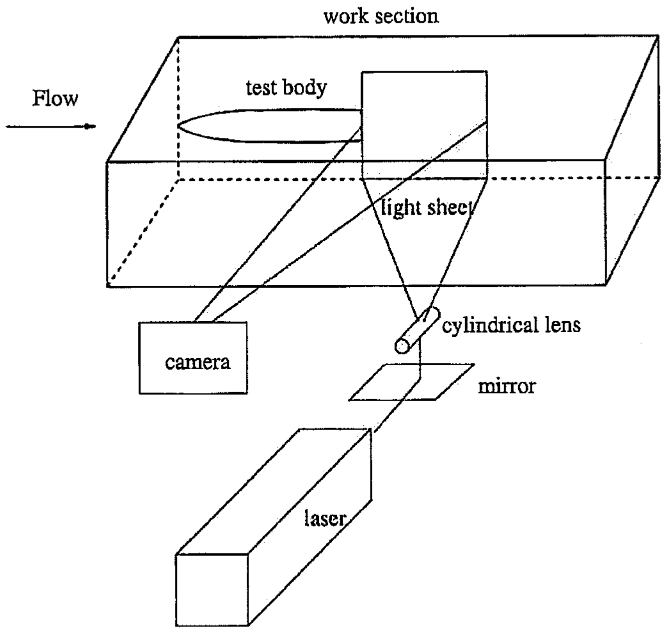


Figure 3: PIV system set-up

(d) The *size of the light sheet* in the region of interest is at least 100 mm wide. Typical thickness are $1-3$ mm. Given that the cylinder is quite small (20 mm), it is advisable to keep the sheet thin, say 1 mm.

The camera image is at least 100×100 mm^2. Hence 0.1 mm per pixel. Assuming an interrogation window of 32×32 pixels, this gives a *spatial resolution* of 3.2 mm. This is quite large so it may be sensible to reduce the interrogation window to 16×16 pixels, giving a *spatial resolution* of 1.6 mm.

Assuming that 20 m/s is the top speed to be measured and ensuring that particles cover half of an interrogation window. We need to set the *interframe time* to

$$\Delta t = \frac{\frac{1}{2} \times 1.6 \times 10^{-3}}{20} = 40\ \mu s \quad (\text{for } 16 \times 16)$$

$$\Delta t = \frac{\frac{1}{2} \times 3.2 \times 10^{-3}}{20} = 80\ \mu s \quad (\text{for } 32 \times 32)$$

The smallest velocity we are likely to be able to resolve is where a particle moves by about 0.5 pixels.

$$V_{min} = \frac{\frac{1}{2} \times 0.1 \times 10^{-3}\ m}{40\ \mu s} = 1.25\ m/s \ (\text{for } 16 \times 16)$$

$$V_{min} = \frac{\frac{1}{2} \times 0.1 \times 10^{-3}\ m}{80\ \mu s} = 0.63\ m/s \text{ (for } 32 \times 32)$$

Dynamic range is equivalent to size of interrogation window , i.e., 16 : 1 for higher resolution or 32 : 1 for larger windows.

(e) Here we need to estimate the size of the boundary layer on the cylinder. If it were laminar, we could use Blasius

$$\delta \approx \frac{x \times 5}{\sqrt{Re_x}} \approx 2 \text{ mm}$$

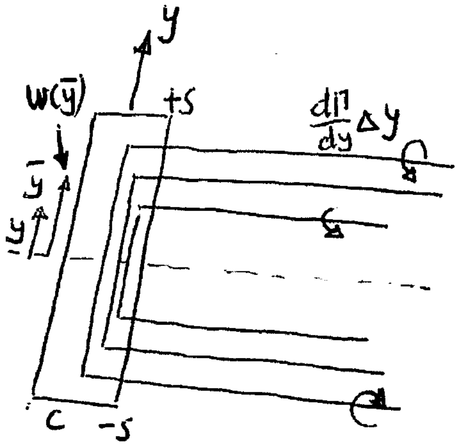
(One could simply argue that on a small object like this, the boundary layer is likely to be just a few *mm*).

Even if turbulence boundary layers are thicker, we would never be able to get more than one or two data points using PIV, so this is not a good method to choose.

Alternatives might include hot wires or LDV, both of which give good spatial resolution. Or, one could set up a new PIV system 'zooming' in on the boundary layer to give higher resolution.

Question 8

(a)



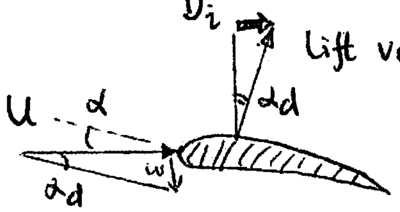
$S = 2sc$

(b) The downwash at spanwise location $\bar{y}$ due to the wake element at y is :

$$dw(\bar{y}) = \underbrace{\frac{1}{2}}_{\text{semi-infinite vortex line}} \cdot \frac{(d\Gamma/dy)\,dy}{2\pi(\bar{y}-y)}$$

Hence: $w(\bar{y}) \cancel{= \frac{1}{4\pi}\int_{-s}^{+s} \frac{d\Gamma}{dy}\frac{dy}{y}} = \frac{1}{4\pi}\int_{-s}^{+s} \frac{d\Gamma}{dy}\frac{dy}{\bar{y}-y}$; $\alpha_d = w(y)/U$

(c)



lift vector, locally $= \rho U \Gamma$

$D_i = (\rho U \Gamma)\sin\alpha_d \simeq (\rho U \Gamma)\alpha_d$.

The effect of the downwash is to rotate the lift vector so there is a streamwise component ; this is the induced drag : $D_i = \rho U \int_{-s}^{+s} \Gamma \alpha \, dy$

Since the local section lift is $\rho U \Gamma$.

(d) Hence, using Data sheet expression for spanwise Circulation distribution.

$$y = -s\cos\theta$$

$$C_L = \frac{\rho U}{\frac{1}{2}\rho U^2 S}\int_{-s}^{s}\Gamma\, dy = \frac{2}{US}\int_0^{\pi}\Big(Us \sum_{n\ odd} G_n \sin n\theta\Big)\, s \sin\theta\, d\theta$$

$$= \frac{2s^2}{S}\int_0^{\pi} G_1 \sin^2\theta\, d\theta \qquad \text{Since } \int_0^{\pi}\sin m\theta \sin n\theta\, d\theta = \begin{cases}\pi/2 & m=n\\ 0 & m\neq n\end{cases} \quad \text{(Given)}$$

$$\therefore\ C_L = \frac{\pi s^2 G_1}{S} = \frac{\pi A\!\!R\, G_1}{4} \quad \text{with } A\!\!R = 4s^2/S.$$

$$C_{D_i} = \frac{\rho U}{\frac{1}{2}\rho U^2 S}\int_{-s}^{+s}\Gamma\, \alpha_i\, dy = \frac{2}{US}\int_0^{\pi}\underbrace{\Big(Us\sum_{n\ odd} G_n \sin\theta\Big)}_{\Gamma}\Big(\frac{1}{4\pi}\sum_{m\ odd} m\, G_m \int_0^{\pi}\frac{\cos m\phi\, d\phi}{\cos\phi - \cos\theta}\Big) \cdots s\sin\theta\, d\theta$$

$$= \frac{2}{US}\int_0^{\pi}\Big(Us\sum_{n\ odd} G_n \sin n\theta\Big)\Big(\frac{1}{4}\sum_{m\ odd} m\, G_m \underbrace{\frac{\sin m\theta}{\sin\theta}}_{\text{(Glauert integral Data sheet)}}\Big)\sin\theta\, d\theta$$

$$= \frac{s^2}{2S}\sum_{n\ odd}\sum_{m\ odd} m\, G_n G_m \underbrace{\int_0^{\pi}\sin m\theta \sin n\theta\, d\theta}_{=\begin{cases}\pi/2 & m=n\\ 0 & m\neq n\end{cases}}$$

$$\therefore\ C_{D_i} = \frac{\pi s^2}{4S}\sum_{n\ odd} n\, G_n^2 = \frac{\pi A\!\!R}{16}\Big(\frac{4C_L}{\pi A\!\!R}\Big)^2 \sum_{n\ odd} n \frac{G_n^2}{G_1^2}$$

$$\therefore\ C_{D_i} = (1+\delta)\frac{C_L^2}{\pi A\!\!R} \quad \text{with } \delta = 3\Big(\frac{G_3}{G_1}\Big)^2 + 5\Big(\frac{G_5}{G_1}\Big)^2 + \cdots$$

Hence the elliptic lift distribution (only G_1 non-zero) has the minumum induced drag.

Numerics Answers

Jie Li

Quesion 3,
(c) Area ratio is $\sqrt{7}$.
Question 5,
(c) $a_0 = 0$, $a_1 = 2$, $a_2 = 0$, $a_3 = -2$ and $a_4 = 1$.
Question 7,
(b) Power factor $\lambda = 0.3$.