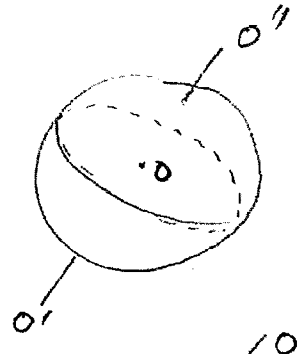
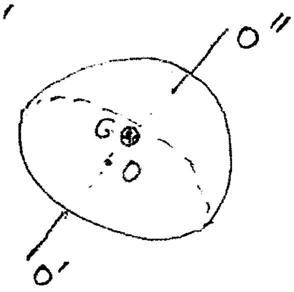
1. (a) From the data book the principal moments of inertia of a cube are $\frac{1}{6} ma^2$. The three are identical hence the cube may be visualized as a sphere.



(b)(i) The prism can then be thought of as a hemisphere. The axis O'O'' shown here is principal both for the sphere as for the hemisphere.

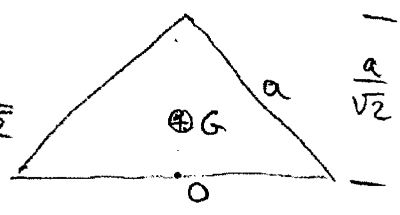


The prism is therefore "axisymmetric" and is therefore an AAC body

with $C = \frac{1}{2}\left(\frac{1}{6} ma^2\right) = \frac{1}{12} ma^2$

To find A we need the distance OG.

We know for a triangle that G is at $\frac{1}{3}$ height so $\overline{OG} = \frac{a}{3\sqrt{2}}$



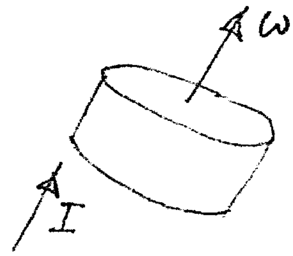
(ii) Parallel axes theorem gives

$$A = \frac{1}{2}\left(\frac{1}{6} ma^2\right) - \frac{m}{2}\left(\overline{OG}\right)^2$$

$$= \frac{1}{12} ma^2 - \frac{m}{2}\,\frac{a^2}{18}$$

$$= \frac{1}{18} ma^2 = 0.0556\, ma^2$$

1. cont (c) Think of the prism as an AAC body – say like a cylinder.



The impulse I is at a distance $\frac{AC}{2}$

$$= \frac{\sqrt{a^2 + (\sqrt{2}a)^2}}{2} = \frac{\sqrt{3}\,a}{2}$$

So the impulsive couple is $\frac{I a \sqrt{3}}{2}$

(i) The impuse causes a change in linear velocity

$$\underset{\sim}{I} = \frac{M}{2}\Delta \underline{v} \quad \therefore \quad \Delta \underline{v} = \frac{2I}{M}$$

in the direction of the impulse

(ii) The frequency of nutation can be obtained from the second of the gyroscope equations

$$A\dot{\Omega}_2 - (C\omega_3 - A\Omega_3)\omega_1 = 0 \quad \text{for free motion}$$

and with Euler angles
$$\Omega_1 = -\dot{\phi}\sin\theta$$
$$\Omega_2 = \dot{\theta}$$
$$\Omega_3 = \dot{\phi}\cos\theta$$

Nutation with small constant θ $\therefore$
$$\Omega_1 \simeq -\dot{\phi}\theta$$
$$\Omega_2 = 0$$
$$\Omega_3 \simeq \dot{\phi}$$

and with spin $\omega_3 \simeq \omega$

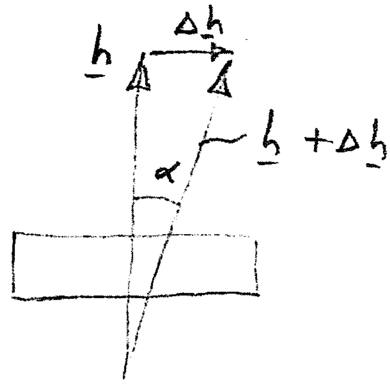
$$C\omega \simeq A\dot{\phi} \quad \therefore \quad \dot{\phi} \simeq \frac{C\omega}{A} = \frac{1}{12}\Big/\frac{1}{18}\,\omega$$

$$= 1{\cdot}5\,\omega$$

1(c)(cont) (iii) For the cone angle consider the change in moment of momentum as a vector

The original moment of momentum $\underset{\sim}{h} = C\omega \underset{\sim}{k}$

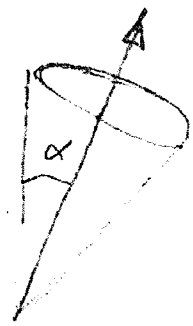
Impulsive couple is $\frac{Ia\sqrt{3}}{2}\underline{i}$



$$\therefore \Delta \underline{h} = \frac{Ia\sqrt{3}}{2}\underline{i}$$

The vector $\underline{h} + \Delta\underline{h}$ is the axis about which the nutation motion occurs

So the semi-cone-angle α



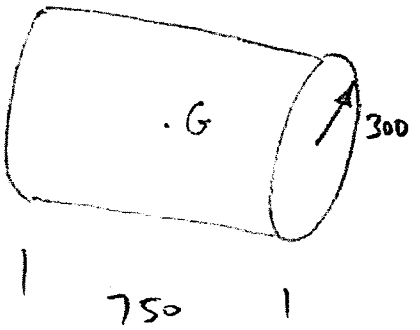
$$\alpha \simeq \frac{|\Delta h|}{|h|}$$

$$= \frac{Ia\sqrt{3}}{2C\omega}$$

$$= \frac{Ia\sqrt{3}}{2\,\frac{1}{12}ma^2\omega}$$

$$\underline{\underline{\alpha = 6\sqrt{3}\,\frac{I}{ma\omega}}}$$

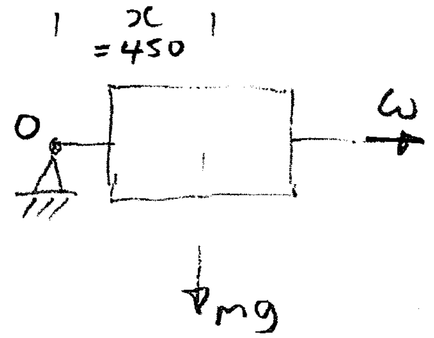
2. Moments of inertia at G: AAC body:



$$C = \frac{1}{2} m a^2$$
$$= \frac{1}{2} \times 340 \times (.3)^2$$
$$= 15.3 \text{ kgm}^2$$

$$A = m\left(\frac{a^2}{4} + \frac{\ell^2}{12}\right) = 340\left(\frac{.3^2}{4} + \frac{.75^2}{12}\right)$$
$$= 23.6 \text{ kgm}^2$$

(a) (i)



Steady precession rate Ω

$$Q = C\omega\Omega$$
$$\therefore \Omega = \frac{Q}{C\omega}$$
$$= \frac{mgx}{C\omega}$$

$$\omega = \frac{2\pi \times 1200}{60} = 125.7 \text{ rad s}^{-1}$$

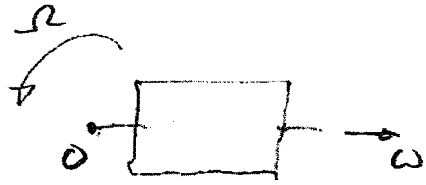
$x = 0.45$ m $\qquad g = 9.81$ ms^{-2}

$$\therefore \Omega = \frac{340 \times 9.81 \times 0.45}{15.3 \times 125.7} = 0.78 \text{ rad s}^{-1}$$

Note that this is independent of the angle of inclination to the horizontal.

2 (a) cont. (ii) Consider moment of momentum about the vertical axis which is zero to start with. Steady precession about O gives rise to moment of momentum

$(A + Mx^2)\,\Omega$



top view

(parallel axis theorem used to get moment of inertia about O)

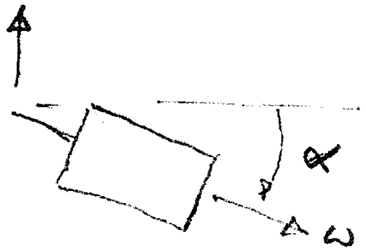
$= (23.6 + 340 \times .45^2) \times 0.78$

$= 72.2 \quad kg\,m^2\,s^{-1}$ ↑

But the moment of momentum about the vertical axis must still be zero hence the spin axis must have tilted downwards

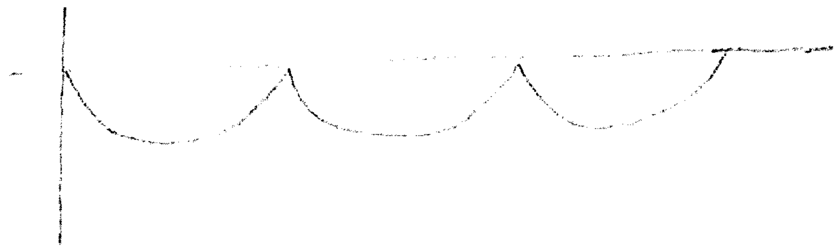
so that $C\omega\alpha = 72.2$

$\therefore \alpha = \dfrac{72.2}{15.3 \times 125.7}$



$= 0.0375$ rad $\equiv 2.15°$

2 cont (b) The motion that occurs after release at one end is "cuspidal nutation"



and the frequency of the nutation is $\frac{Cw}{(A + Mx^2)}$

$$= \frac{15.3 \times 125.7}{(23.6 + 340 \times 0.45^2)} = 20.8 \text{ rad s}^{-1}$$

$$\equiv 3.31 \text{ Hz}$$

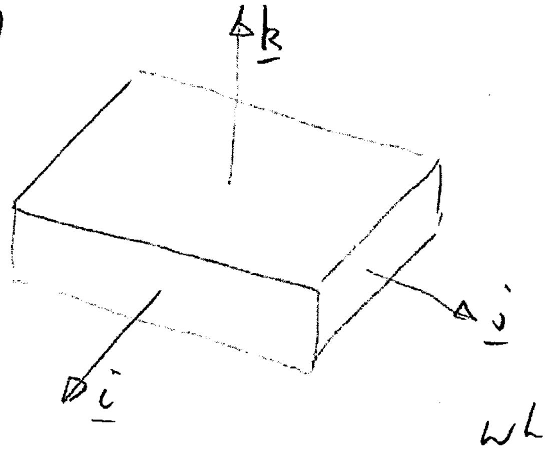
The period of cuspidal motion is therefore $\frac{1}{3.31}$

$= 0.302$ seconds

(i) A release time $T = 0.01$ s is very short compared with the nutation period so release will be "clean"

(ii) The worst point to release would be at the bottom of the cuspidal motion ie $T = 0.15$ seconds.

3. (a)



$$\underline{h} = A\omega_1 \underline{i} + B\omega_2 \underline{j} + C\omega_3 \underline{k}$$

$$\dot{\underline{h}} = \dot{\underline{h}}\big|_{rot} + \underline{\omega} \times \underline{h}$$

where $\dot{\underline{h}}\big|_{rot}$ is the rate of change of moment of momentum calculated as if the reference frame is not rotating. $\underline{\omega}$ is the angular velocity of the reference frame which, for a body-fixed frame, is equal to the rate of rotation of the body.

$$\underline{\omega} \times \underline{h} = (\omega_1 \underline{i} + \omega_2 \underline{j} + \omega_3 \underline{k}) \times$$
$$\times (A\omega_1 \underline{i} + B\omega_2 \underline{j} + C\omega_3 \underline{k})$$

$$\begin{aligned} = \; & (\omega_2 C\omega_3 - \omega_3 B\omega_2)\, \underline{i} \\ & + (\omega_3 A\omega_1 - \omega_1 C\omega_3)\, \underline{j} \\ & + (\omega_1 B\omega_2 - \omega_2 A\omega_1)\, \underline{k} \end{aligned}$$

and with $\underline{Q} = Q_1 \underline{i} + Q_2 \underline{j} + Q_3 \underline{k} = \dot{\underline{h}}$

$$\therefore \left.\begin{aligned} Q_1 &= A\dot{\omega}_1 - (B - C)\,\omega_2\omega_3 \\ Q_2 &= B\dot{\omega}_2 - (C - A)\,\omega_3\omega_1 \\ Q_3 &= C\dot{\omega}_3 - (A - B)\,\omega_1\omega_2 \end{aligned}\right\} \text{Euler's Equations}$$

3. cont (b) Free motion in space $\therefore Q_1 = Q_2 = Q_3 = 0$

and for $\underline{\omega} = \Omega \underline{i}$ $\quad\therefore$

$$\omega_1 = \Omega \qquad \dot{\omega}_1 = 0$$
$$\omega_2 = 0 \qquad \dot{\omega}_2 = 0$$
$$\omega_3 = 0 \qquad \dot{\omega}_3 = 0$$

$\therefore$ Euler's equations are satisfied in steady state.

Perturb the motion so that

$$\omega_1 = \Omega + \omega_1' \qquad \dot{\omega}_1 = \dot{\omega}_1'$$
$$\omega_2 = \omega_2' \qquad \dot{\omega}_2 = \dot{\omega}_2'$$
$$\omega_3 = \omega_3' \qquad \omega_3 = \dot{\omega}_3'$$

with small ω_1' ω_2' ω_3' $\dot{\omega}_1'$ $\dot{\omega}_2'$ $\dot{\omega}_3'$

Euler's equations :

$$0 = A\dot{\omega}_1' - (B-C)\,\omega_2'\,\omega_3'$$
$$0 = B\dot{\omega}_2' - (C-A)\,\omega_3'\,(\Omega + \omega_1')$$
$$0 = C\dot{\omega}_3' - (A-B)(\Omega + \omega_1')\,\omega_2'$$

ignore small quantities $\omega_2'\omega_3'$, $\omega_3'\omega_1'$ & $\omega_1'\omega_2'$

$\therefore \quad A\dot{\omega}_1' \simeq 0 \qquad \therefore \quad \omega_1' \simeq \text{constant}$

$$\left.\begin{aligned} & B\dot{\omega}_2' - (C-A)\,\Omega\,\omega_3' \simeq 0 \quad ① \\ & C\dot{\omega}_3' - (A-B)\,\Omega\,\omega_2' \simeq 0 \quad ② \end{aligned}\right\} \quad \&$$

$$\therefore \left.\begin{aligned} & \ddot{\omega}_2' - \frac{C-A}{B}\,\Omega\,\dot{\omega}_3' \simeq 0 \quad (1^a) \\ & \dot{\omega}_3' \simeq \frac{A-B}{C}\,\Omega\,\omega_2' \quad (1^b) \end{aligned}\right\}$$

Substitute (1^b) into (1^a)

$$\therefore \quad \ddot{\omega}_2' + \frac{(A-C)(A-B)}{BC}\,\Omega^2\,\omega_2' \simeq 0$$

3(b) cont. $\therefore \ddot{\omega}_2' + \lambda^2 \omega_2' \simeq 0$

$$\lambda^2 = \frac{(A-C)(A-B)}{BC}\,\Omega^2$$

This is SHM for $\lambda^2 > 0$ ($\sin\lambda t$ solutions)

ie $A > C$ & $A > B$

or $A < C$ & $A < B$

and unstable motion for $\lambda^2 < 0$ ($e^{\lambda t}$ solutions)

ie $A < C$ & $A > B$ ie $B < A < C$

or $A > C$ & $A < B$ ie $C < A < B$

(c) The body shown is a lamina

so $C \simeq A + B$ $\therefore C > A$

and $C > B$

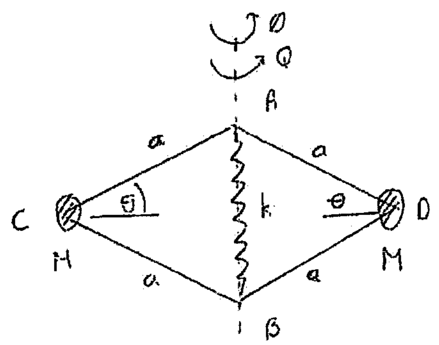
Motion about $\underline{k}$ is stable

The diagram suggests that $A > B$

so motion about $\underline{i}$ is unstable

motion about $\underline{j}$ is stable

1. a)



Velocity of C $\longleftarrow$

$\frac{d}{dt}(a\cos\theta)$

$= a\dot{\theta}\sin\theta$

$\longleftrightarrow$ $a\cos\theta$

Velocity of D $= a\dot{\theta}\sin\theta$ $\longrightarrow$

$\otimes$ Velocity of D into page $= \dot{\phi}\, a\cos\theta$

$\odot$ Velocity of C out of page $= \dot{\phi}\, a\cos\theta$

Resultant velocity of C: $(a^2\dot{\theta}^2\sin^2\theta + a^2\dot{\phi}^2\cos^2\theta)^{1/2}$

Resultant velocity of D = Velocity of C

$\Rightarrow$ <u>Kinetic energy</u> $T = \frac{1}{2}M \times 2 \times [a^2\dot{\theta}^2\sin^2\theta + a^2\dot{\phi}^2\cos^2\theta]$

Length of spring $= 2a\sin\theta \Rightarrow$ extension $= 2a(\sin\theta - \sin\theta_0)$

$\Rightarrow$ <u>Potential energy</u> $U = \frac{1}{2}k \times 4a^2(\sin\theta - \sin\theta_0)^2$

<u>Lagrange for θ</u> $\quad \frac{\partial T}{\partial \dot{\theta}} = 2Ma^2\dot{\theta}\sin^2\theta \quad \frac{\partial T}{\partial \theta} = 2Ma^2\sin\theta\cos\theta\,[\dot{\theta}^2 - \dot{\phi}^2]$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\theta}}\right] - \frac{\partial T}{\partial \theta} + \frac{\partial U}{\partial \theta} = 0$$

$$\Rightarrow 2Ma^2\ddot{\theta}\sin^2\theta + 4Ma^2\dot{\theta}^2\sin\theta\cos\theta - 2Ma^2\sin\theta\cos\theta\,[\dot{\theta}^2 - \dot{\phi}^2] + 4ka^2\cos\theta(\sin\theta - \sin\theta_0) = 0$$

$$\underline{M\ddot{\theta}\sin^2\theta + M\dot{\theta}^2\sin\theta\cos\theta + M\dot{\phi}^2\sin\theta\cos\theta + 2k\cos\theta(\sin\theta - \sin\theta_0) = 0} \quad \text{—— (1)}$$

<u>Lagrange for φ</u> $\quad \frac{\partial T}{\partial \dot{\phi}} = 2Ma^2\cos^2\theta\,.\,\dot{\phi}$

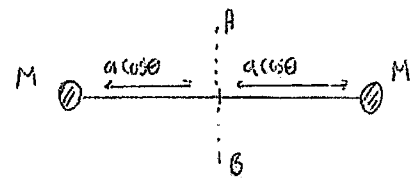
$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\phi}}\right] - \frac{\partial T}{\partial \phi} + \frac{\partial U}{\partial \phi} = Q$$

$\uparrow$ generalised force in φ

$$\underline{\frac{d}{dt}[2Ma^2\cos^2\theta\,.\,\dot{\phi}] = Q}$$

$\uparrow$

generalised momentum in φ = moment of momentum around AB



[40%]

b) For $\dot{\phi} = \Omega$ and steady solution $\dot{\theta} = \ddot{\theta} = 0$, equation ① becomes:

$$M\Omega^2 \sin\theta\cos\theta + 2k\cos\theta(\sin\theta - \sin\theta_0) = 0$$

$$\Rightarrow \quad \underline{\sin\theta_e = \frac{2k\sin\theta_0}{M\Omega^2 + 2k}}$$

[20%]

c) For $\Omega = 0$, $\theta_e = \theta_0$. Consider equation ①:

$$\sin\theta = \sin((\theta-\theta_0)+\theta_0) = \sin(\theta-\theta_0)\cos\theta_0 + \cos(\theta-\theta_0)\sin\theta_0$$

$$\underbrace{M\ddot{\theta}\sin^2\theta}_{\text{Approx } \sin^2\theta_0} + \underbrace{M\dot{\theta}^2\sin\theta\cos\theta}_{\text{small} \approx 0} + \underbrace{M\dot{\phi}^2\sin\theta\cos\theta}_{0} + 2k\cos\theta(\sin\theta - \sin\theta_0) = 0$$

$$\cos\theta = \cos((\theta-\theta_0)+\theta_0)$$

$$= \cos(\theta-\theta_0)\cos\theta_0 - \sin(\theta-\theta_0)\sin\theta_0$$

This term becomes: $2k(\theta-\theta_0)\cos^2\theta_0$

Put $\hat{\theta} = \theta - \theta_0 \Rightarrow \quad M\ddot{\hat{\theta}}\sin^2\theta_0 + 2k\cos^2\theta_0\hat{\theta} = 0$

$$\Rightarrow \quad \omega_n^2 = \frac{2k\cos^2\theta_0}{M\sin^2\theta_0} = \underline{\frac{2k}{M\tan^2\theta_0}}$$

[40%]

For $\Omega \neq 0$ need to expand around $\theta = \theta_e$ and include $\dot{\phi}^2$ term.

2 a) $$T = \sum_n \sum_m \{ A_{nm} \dot{q}_n \dot{q}_m + B_{nm} \dot{q}_n q_m + C_{nm} q_n q_m \}$$

$$V = \sum_n \sum_m D_{nm} q_n q_m$$

Lagrange: $$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_j}\right] - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = Q_j$$

$$\frac{\partial T}{\partial \dot{q}_j} = \underbrace{\sum_n A_{nj} \dot{q}_n}_{m=j} + \underbrace{\sum_m A_{jm} \dot{q}_m}_{n=j} + \underbrace{\sum_m B_{jm} q_m}_{n=j}$$

$A_{nm} = A_{mn} \Rightarrow$ $$\frac{\partial T}{\partial \dot{q}_j} = 2 \sum_n A_{jn} \dot{q}_n + \sum_n B_{jn} q_n$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_j}\right] = 2 \sum_n A_{jn} \ddot{q}_n + \sum_n B_{jn} \dot{q}_n \qquad \text{———} \quad (1)$$

Also, $$\frac{\partial T}{\partial q_j} = \underbrace{\sum_n B_{nj} \dot{q}_n}_{m=j} + \underbrace{2 \sum_n C_{jn} q_n}_{\text{combination of } n=j \text{ and } m=j \text{ and use of symmetry}} \qquad \text{———} \quad (2)$$

Similarly $$\frac{\partial V}{\partial q_j} = \underbrace{2 \sum_n D_{jn} q_n}_{\text{combination of } n=j \text{ and } m=j \text{ and symmetry}} \qquad \text{———} \quad (3)$$

So: (1), (2), (3) $\Rightarrow$ $$2 \sum_n A_{jn} \ddot{q}_n + \sum_n B_{jn} \dot{q}_n - \sum_n B_{nj} \dot{q}_n - 2 \sum_n C_{jn} q_n + 2 \sum_n D_{jn} q_n = Q_j$$

Put $\underline{M_{jn} = 2A_{jn}}$, $\underline{L_{jn} = B_{jn} - B_{nj}}$, $\underline{k_{jn} = 2D_{jn} - 2C_{jn}}$

$$\Rightarrow \quad \underset{\wedge}{M} \ddot{\underset{\wedge}{q}} + \underset{\wedge}{L} \dot{q} + \underset{\wedge}{k} \underset{\wedge}{q} = \underset{\wedge}{Q}$$ [0%]

b) M and k symmetric since A, D and C are symmetric

$L_{jn} = B_{jn} - B_{nj} \Rightarrow L_{nj} = B_{nj} - B_{jn} = -L_{jn} \Rightarrow$ $\underline{L \text{ skew-symmetric}}$ [20%]

c) Power = Force × velocity.

For coordinate j, force due to gyroscopic damping = $\sum_n L_{jn} \dot{q}_n$

$\Rightarrow$ Power = Force × velocity = $\dot{q}_j \times \sum_n L_{jn} \dot{q}_n$

Summing the power over all coordinates $P = \sum_j \sum_n L_{jn} \dot{q}_j \dot{q}_n = \dot{\underset{\frown}{q}}^T L \dot{\underset{\frown}{q}}$

Now $P = \sum_j \sum_n L_{jn} \dot{q}_j \dot{q}_n = \frac{1}{2} \sum_j \sum_n \underbrace{(L_{jn} + L_{nj})} \dot{q}_j \dot{q}_n$

but $L_{jn} = -L_{nj}$ (skew symmetric).

$\Rightarrow \underline{P = 0}$

For a conventional damping matrix $L_{jn} + L_{nj} \neq 0 \Rightarrow P \neq 0$ [30%]