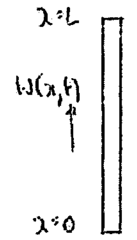
1 a) Differential equation:

$$\rho A \frac{\partial^2 w}{\partial t^2} - EA \frac{\partial^2 w}{\partial x^2} = 0$$




Free end $\Rightarrow$ $\underline{EAw'(L)=0}$

Free end $\Rightarrow$ $\underline{EAw'(0)=0}$

Assume $w = e^{-ikx + i\omega t}$

$$\Rightarrow \quad \rho A\omega^2 = EAk^2 \Rightarrow \omega = \sqrt{\frac{E}{\rho}}\, k$$

Soln has the form $w(x,t) = e^{i\omega t}(A_1 \cos kx + A_2 \sin kx)$

$$w'(0,t) = 0 \Rightarrow A_2 = 0$$

$$w'(L,t) = 0 \Rightarrow \underline{\sin kL = 0} \Rightarrow kL = n\pi, \; n = 0,1,2..$$

$$\Rightarrow \underline{\omega_n = \sqrt{\frac{E}{\rho}}\, k_n = \left(\frac{n\pi}{L}\right)\sqrt{E/\rho}}$$ [30%]

b) Mode shape $u_n(x) = C \cos k_n x$
↑ normalisation constant

$$\int \rho u_n^2(x)\,dx = 1 \Rightarrow \rho C^2 \int_0^L \cos^2\left(\frac{\pi n}{L}\right)x\,dx = \tfrac{1}{2}\rho L C^2 = 1 \Rightarrow \underline{C = \sqrt{\frac{2}{\rho L A}}} \quad n = 1,2,3$$

$\underline{\text{unless}}$ $n=0$, when $\rho C^2 \int_0^L 1\,dx = \rho L C^2 = 1 \Rightarrow \underline{C = \sqrt{\frac{1}{\rho L A}}}$ $n=0$

$$\Rightarrow \underline{u_0(x) = \sqrt{\frac{1}{\rho L A}}} \; ; \; \underline{u_n(x) = \sqrt{\frac{2}{\rho L A}} \cos k_n x} \quad n = 1,2,3..$$ [20%]

c) From data sheet

$$F\,h(L,0,t) = \sum_n \frac{u_n(0)u_n(L)}{\omega_n^2}\left[1 - \cos\omega_n t\right] F$$

but, what about $\omega_n = 0$? $\quad \frac{1-\cos\omega_n t}{\omega_n^2} \longrightarrow \frac{1-(1-\frac{1}{2}\omega_n^2 t^2)}{\omega_n^2} = \frac{1}{2}t^2$

or use L'Hopital's rule $\quad \lim_{\omega_n \to 0}\left\{ \frac{t \sin\omega_n t}{2\omega_n} \right\} \longrightarrow \frac{1}{2}t^2$

1 Cont'd

Contribution from rigid body mode $= \frac{F u_0(0) u_0(L)}{\omega_0^2}[1-\cos\omega_0 t] = \frac{1}{2}t^2 u_0(0)u_0(L)$

$= \frac{1}{2}t^2\left(\frac{F}{\rho L A}\right)$

↑

this is just $\frac{F}{M}$ = acceleration of the launcher as a rigid body

"Displacement $= \frac{1}{2}at^2$" for constant acceleration.

So $\underline{F h(L,0,t) = \frac{1}{2}t^2\left(\frac{F}{\rho L A}\right) + \sum_{n=1}^{\infty} \frac{u_n(0)u_n(L)}{\omega_n^2}[1-\cos\omega_n t] F}$ [30%]

d) $\omega_n = \sqrt{\frac{E}{\rho}}\left(\frac{n\pi}{L}\right) = 5000 \times \frac{n\pi}{50} = n \times 314.2$ rads/sec

↑

Speed of sound

$= \frac{n \times 100\pi}{2\pi} = \underline{50n \text{ Hz}}$

Maximum response at $x=L$ due to mode $n$ is proportional to $\frac{1}{\omega_n^2}$

To avoid resonant excitation arising from mode 1 response, need satellite natural frequencies to be $\gg \omega_1$. Given that $\omega_1 = 50$Hz, having natural frequencies $> 100$Hz is reasonable. [20%]

2 a) For a string $\underline{\omega_n = \left(\frac{n\pi}{L}\right)\sqrt{\frac{P}{m}}}$

$$\underline{u_n(x) = C \sin\left(\frac{n\pi}{L}\right)x}$$

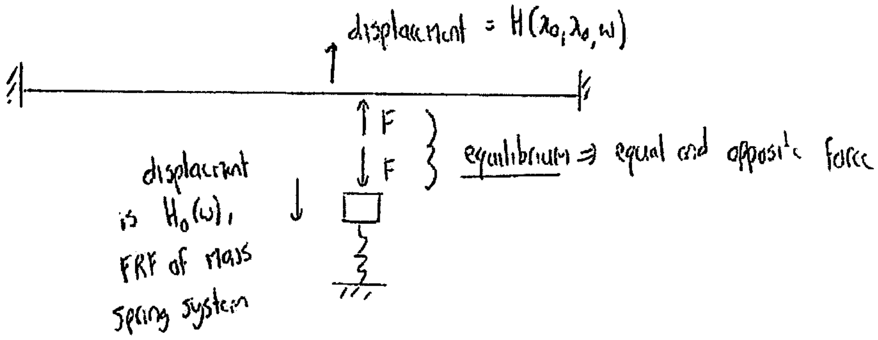
$\uparrow$ normalisation constant

$$\int_0^L m\, u_n^2(x)\,dx = 1 \quad \Rightarrow \quad \underline{C = \sqrt{\frac{2}{mL}}}$$ [15%]

b) From the datasheet $H(x_0, x_0, \omega) = \sum_n \frac{u_n(x_0)\,u_n(x_0)}{\omega_n^2 - \omega^2}$

$$\underline{H(x_0, x_0, \omega) = \left(\frac{2}{mL}\right) \sum_n \frac{\sin^2\left(\frac{n\pi}{L}\right)x_0}{\omega_n^2 - \omega^2}}$$ [15%]

c) Consider compatibility and equilibrium conditions




$$H_0(\omega) = \frac{u(x_0)\,u(x_0)}{\omega_0^2 - \omega^2}$$ ; $u(x_0)$ = mass/spring mode shape scaled to unit generalised mass

$$\Rightarrow u(x_0) = \frac{1}{\sqrt{M}}$$

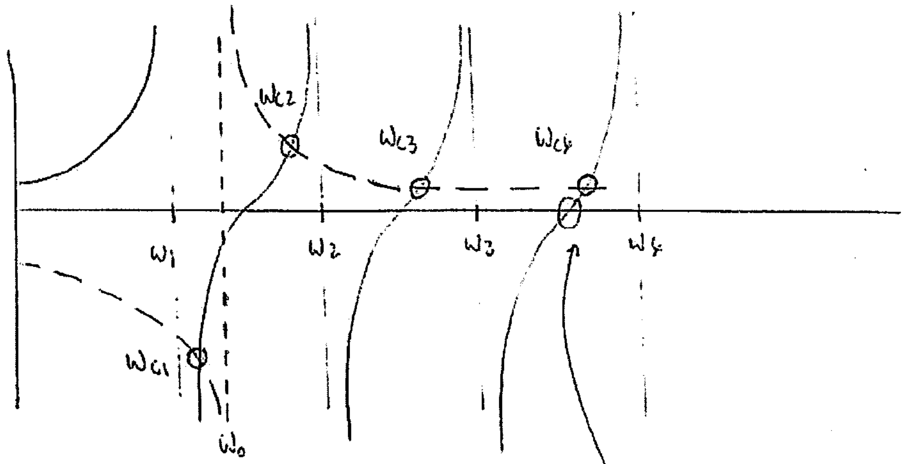
$$\Rightarrow H_0(\omega) = \frac{1}{M(\omega_0^2 - \omega^2)}$$

For $\underline{\text{compatibility}}$ $H(x_0, x_0, \omega) = -H_0(\omega)$

$$\Rightarrow \underline{H(x_0, x_0, \omega) = \frac{1}{M(\omega^2 - \omega_0^2)}}$$ [25%]

2 cont

d)

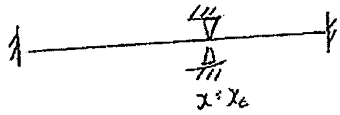



note that zero's of $H(x_0, x_0, \omega)$ occur at frequencies corresponding to the resonances of a system with a clamp at '$x_0$'

key: ——— $H(x_0, x_0, \omega)$

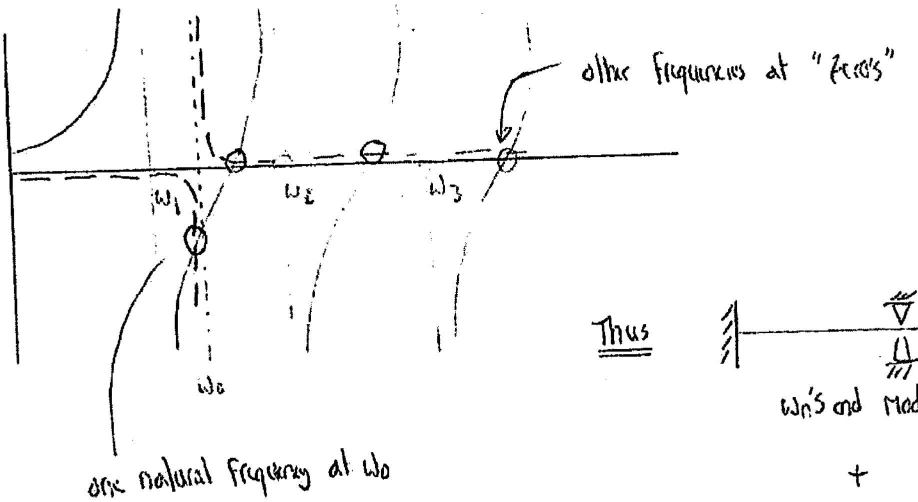
– – – – $\dfrac{1}{M(\omega^2 - \omega_0^2)}$

o natural frequencies of coupled system

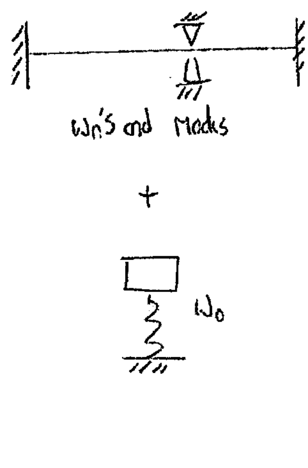



By inspection $\omega_{cj}$ is closer to $\omega_0$ than is $\omega_j$ [25%]

e) For $M \to \infty$, $\dfrac{1}{M(\omega^2 - \omega_0^2)}$ is close to zero apart from the immediate vicinity of $\omega_0$:

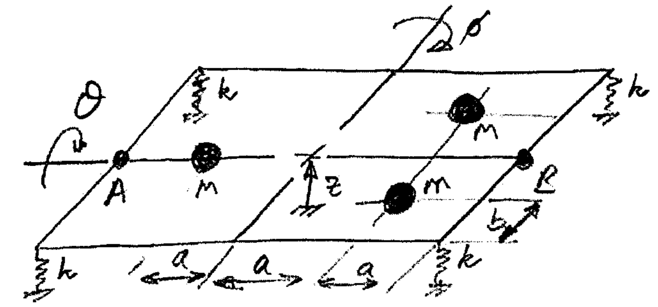



Thus




[20%]

3.




(a) $\underline{\underline{KE}}$ $T = \frac{1}{2}m\left[\dot{z} + a\dot{\phi}\right]^2 + \frac{1}{2}m\left[\dot{z} - a\dot{\phi} - b\dot{\theta}\right]^2 + \frac{1}{2}m\left[\dot{z} - a\dot{\phi} + b\dot{\theta}\right]^2$

Lagrange: $\partial T/\partial \dot{z} = m[\dot{z} + \cancel{a\dot{\phi}} + \dot{z} - \cancel{a\dot{\phi}} - \cancel{b\dot{\theta}} + \dot{z} - a\dot{\phi} + \cancel{b\dot{\theta}}] \Rightarrow \frac{d}{dt}\left(\partial T/\partial \dot{z}\right) = m(3\ddot{z} - a\ddot{\phi})$

$\partial T/\partial \dot{\phi} = ma[\cancel{\dot{z}} + a\dot{\phi} - \cancel{\dot{z}} + a\dot{\phi} + \cancel{b\dot{\theta}} - \dot{z} + a\dot{\phi} - \cancel{b\dot{\theta}}] \Rightarrow \frac{d}{dt}\left(\partial T/\partial \dot{\phi}\right) = ma(3a\ddot{\phi} - \ddot{z})$

$\partial T/\partial \dot{\theta} = mb[-\cancel{\dot{z}} + \cancel{a\dot{\phi}} + b\dot{\theta} + \cancel{\dot{z}} - \cancel{a\dot{\phi}} + b\dot{\theta}] \Rightarrow \quad \frac{d}{dt}\left(\partial T/\partial \dot{\theta}\right) = mb\,(2b\ddot{\theta})$

$$\Rightarrow \frac{d}{dt}\left(\partial T/\partial \dot{q}\right) = m\begin{bmatrix} 3 & 0 & -a \\ 0 & 2b^2 & 0 \\ -a & 0 & 3a^2 \end{bmatrix}\begin{Bmatrix} \ddot{z} \\ \ddot{\theta} \\ \ddot{\phi} \end{Bmatrix} \quad //$$

$\underline{\underline{PE}}$ $V = \frac{1}{2}k\left\{[z + 2a\phi + 2b\theta]^2 + [z + 2a\phi - 2b\theta]^2 + [z - 2a\phi + 2b\theta]^2 + [z - 2a\phi - 2b\theta]^2\right\}$

$\partial V/\partial z = k(4z)$, $\partial V/\partial \phi = k\,2a\,[8a\phi]$, $\partial V/\partial \theta = k\,2b\,[8b\theta]$

$$\Rightarrow \partial V/\partial q = k\begin{bmatrix} 4 & 0 & 0 \\ 0 & 16b^2 & 0 \\ 0 & 0 & 16a^2 \end{bmatrix}\begin{Bmatrix} z \\ \theta \\ \phi \end{Bmatrix} \quad //$$

[Alternatively write $T = \frac{1}{2}\ddot{\underset{\sim}{q}}^T[M]\ddot{\underset{\sim}{q}}$ & $V = \frac{1}{2}\underset{\sim}{q}^T[K]\underset{\sim}{q}$
and thereby deduce $[M]$ & $[K]$ ] $\Rightarrow [M]\ddot{\underset{\sim}{q}} + [K]\underset{\sim}{q} = \underset{\sim}{0}$

(b) The second row is an equation in $\theta$ only – ie $\theta$ is de-coupled from $z$ & $\phi$:

$$2b^2m\ddot{\theta} + 16b^2k\theta = 0 \quad \Rightarrow \quad \omega = \sqrt{\frac{16\cancel{b^2}k}{2\cancel{b^2}m}} = \underline{\underline{\sqrt{8k/m}}}$$

This corresponds to rotation about the longitudinal axis

For the other two modes, find $\omega^2$ from $\det[K - \omega^2 M] = \underset{\sim}{0}$

$$\Rightarrow \begin{vmatrix} 4k - 3\omega^2 m & -ma\omega^2 \\ -ma\omega^2 & 16ka^2 - 3ma^2\omega^2 \end{vmatrix} = 0$$

3 cont

$$\Rightarrow (4k - 3\omega^2 m)(16ka^2 - 3ma^2\omega^2) - m^2a^2\omega^4 = 0$$

$$\Rightarrow 64k^2a^2 + (-12-48)\omega^2 mka^2 + 8m^2a^2\omega^4 = 0$$

$$\div k^2a^2 \Rightarrow 8\lambda^2 - 60\lambda + 64 = 0 \qquad \text{where } \lambda = \omega^2 m/k$$

$$\Rightarrow 2\lambda^2 - 15\lambda + 16 = 0 \Rightarrow \lambda = \frac{15 \pm \sqrt{225-128}}{4} = \frac{15}{4} \pm \frac{\sqrt{97}}{4}$$

$$\therefore \lambda = 1.29,\ 6.21$$

$$\Rightarrow \omega = 1.14\sqrt{\frac{k}{m}},\ 2.49\sqrt{\frac{k}{m}} \quad //$$

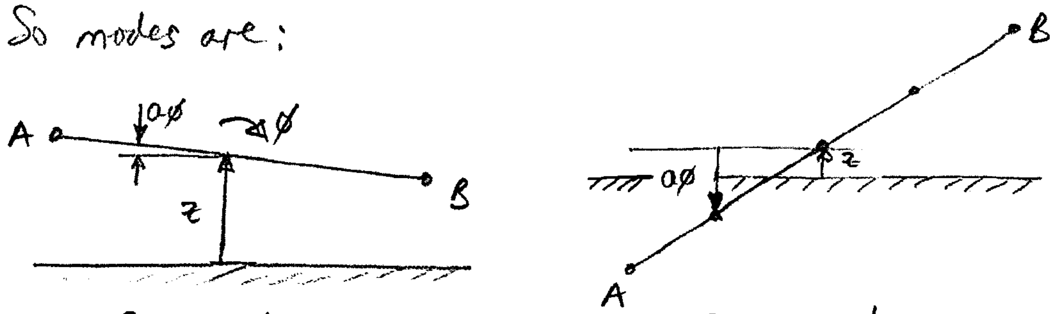
Mode shapes: $[K]\underset{\sim}{\phi} - \omega^2[m]\underset{\sim}{\phi} = 0$

$$\Rightarrow (4k - 3\omega^2 m)z - ma\omega^2\phi = 0$$

$$\text{ie } (4-3\lambda)z - a\lambda\phi = 0$$

$$\Rightarrow \frac{z}{a\phi} = \frac{\lambda}{4-3\lambda} = 9.9,\ -0.42$$

So modes are:

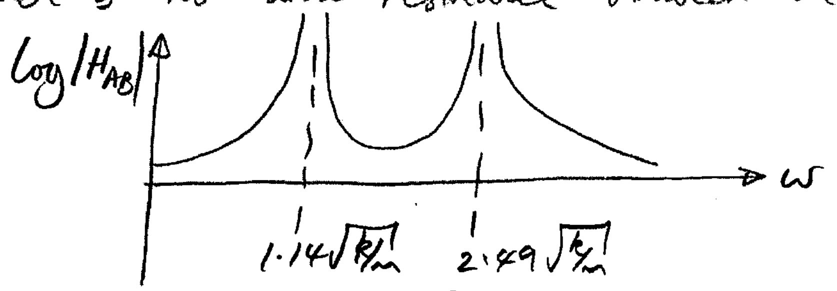



$\omega^2 = 1.29\,k/m$ $\qquad$ $\omega^2 = 6.21\,k/m$

$z = 9.9a\phi$ $\qquad$ $z = -0.42a\phi \rightarrow a\phi = -2.4z$

(c) • $\theta$ mode doesn't show on transfer function for $H_{AB}$, since both points lie on node line $\Rightarrow$ only 2 resonances.

• Product of mode shapes at A & B is positive for lower mode & negative for higher mode. Since sign reverses, there is no anti resonance between the 2 resonances



4 (a) $T = \frac{1}{2}J\dot{\theta}_1^2 + \frac{1}{2}J\dot{\theta}_2^2 + \frac{1}{2}J\dot{\theta}_3^2 + \frac{1}{2}J\dot{\theta}_4^2 + \frac{1}{2}J\dot{\theta}_5^2$

$V = \frac{1}{2}k(\theta_2-\theta_1)^2 + \frac{1}{2}k(\theta_3-\theta_2)^2 + \frac{1}{2}k(\theta_4-\theta_3)^2 + \frac{1}{2}s(\theta_5-\theta_4)^2$

(b) $\omega^2 \simeq R = \frac{V}{T^*} = \frac{\frac{1}{2}k[(\theta_2-\theta_1)^2 + (\theta_3-\theta_2)^2 + (\theta_4-\theta_3)^2] + \frac{1}{2}s(\theta_5-\theta_4)^2}{\frac{1}{2}J[\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_5^2]}$ —— ①

Lowest frequency is for rigid body mode $\underline{q} = [1\ 1\ 1\ 1\ 1]^T$

① $\Rightarrow$ $\omega^2 = \frac{\frac{1}{2}k[0^2] + \frac{1}{2}s[0^2]}{\frac{1}{2}J[5]} = \underline{\underline{0}}$ ✓

(c) When $s=k$, the system is symmetric.

For $q = [1\ \alpha\ 0\ -\alpha\ -1]^T$:

$$\omega^2 \simeq R = \frac{\cancel{\frac{1}{2}}k[(\alpha-1)^2 + (0-\alpha)^2 + (-\alpha-0)^2 + (-1+\alpha)^2]}{\cancel{\frac{1}{2}}J[1^2+\alpha^2+0^2+\alpha^2+1^2]}$$

$$= \frac{k}{J}\frac{\cancel{2}[(1-\alpha)^2+\alpha^2]}{\cancel{2}[1+\alpha^2]} = \frac{k}{J}\frac{(2\alpha^2-2\alpha+1)}{1+\alpha^2} \quad \text{—— ②}$$

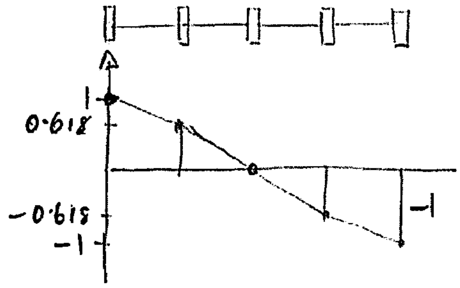
The exact mode shape is found by minimizing $R$:

$$\frac{dR}{d\alpha} = \frac{(1+\alpha^2)(4\alpha-2) - (2\alpha^2-2\alpha+1)(2\alpha)}{(1+\alpha^2)^2} = 0$$

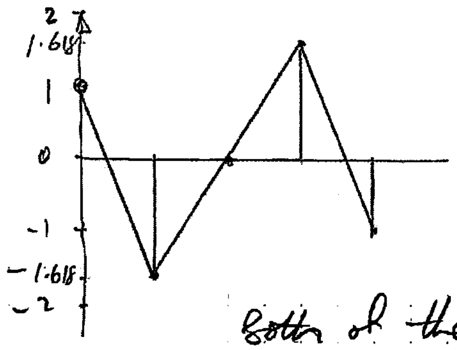
$\Rightarrow \quad \alpha^2 + \alpha - 1 = 0$

$\Rightarrow \quad \alpha = -\frac{1}{2} \pm \frac{\sqrt{5}}{2} = 0.618 \ \& \ -1.618$

& ② $\Rightarrow \quad \omega_1^2 = 0.3819\ k/J \quad \& \quad \omega_2^2 = 2.618\ k/J$




Mode 1, $\alpha = 0.618$, $\omega_1 = 0.618\sqrt{k/J}$




Mode 2, $\alpha = -1.618$, $\omega_2 = 1.618\sqrt{k/J}$

Both of these modes satisfy the mode shape $[1\ \alpha\ 0\ -\alpha\ -1]^T$ and they are therefore both exact

4 (cont) (d)

For a small change in $s$, ie $s = 1{\cdot}1k$, the original mode shape is still a reasonable approximation to the correct mode. Rayleigh will therefore give a reasonable estimate of the frequency:

$$(1) \Rightarrow \quad \omega^2 \simeq \frac{\frac{1}{2}k\left[(\alpha-1)^2 + (0-\alpha)^2 + (-\alpha-0)^2\right] + \frac{1}{2}(1{\cdot}1k)(-1+\alpha)^2}{\frac{1}{2}J\left[2(1+\alpha^2)\right]}$$

$$= \underbrace{\frac{\frac{1}{2}k\left[(\alpha-1)^2 + 2\alpha^2 + (\alpha-1)^2\right]}{\frac{1}{2}J\left[2(1+\alpha^2)\right]}}_{\omega_1^2} + \frac{\cancel{\tfrac{1}{2}}(0{\cdot}1k)(\alpha-1)^2}{\cancel{\tfrac{1}{2}}J\left[2(1+\alpha^2)\right]}$$

$$= \omega_1^2 + \underbrace{\frac{0{\cdot}1k(\alpha-1)^2}{2J(1+\alpha^2)}}_{\Delta\omega^2}$$

For $\alpha = 0{\cdot}618$, $\quad \Delta\omega^2 = 0{\cdot}00528\ k/J$

So $\omega^2 = (0{\cdot}3819 + 0{\cdot}00528)\ k/J = 0{\cdot}3872\ k/J$

Hence % increase in $\omega_1$ is $\dfrac{\sqrt{0{\cdot}3872} - 0{\cdot}618}{0{\cdot}618} = \underline{\underline{0{\cdot}686\%}}$

ENGINEERING TRIPOS PART IIA

Module 3C6 Examination, 2011

Answers

1. (b) $u_0(x) = \sqrt{\dfrac{1}{\rho LA}}; \qquad u_n(x) = \sqrt{\dfrac{2}{\rho LA}}\cos\dfrac{n\pi x}{L}, \quad n = 1,2,3\ldots$

(c) $Fh(L,0,t) = \dfrac{t^2}{2}\left(\dfrac{F}{\rho AL}\right) + F\sum_{n=1}^{\infty}\dfrac{u_n(0)u_n(L)}{\omega_n^2}(1-\cos\omega_n t)$ (d) $f_n = 50n \quad Hz.$

2. (a) $\omega_n = \left(\dfrac{n\pi}{L}\right)\sqrt{\dfrac{P}{m}}; \qquad u_n(x) = \sqrt{\dfrac{2}{mL}}\sin\dfrac{n\pi x}{L}, \quad n = 1,2,3\ldots$

(b) $H(x_o, x_o, \omega) = \left(\dfrac{2}{mL}\right)\sum_{n=1}^{\infty}\dfrac{\sin^2(\frac{n\pi x_o}{L})}{\omega_n^2 - \omega^2}$

3. (a)
$$V = \frac{1}{2}k\left\{[z+2a\phi+2b\theta]^2 + [z+2a\phi-2b\theta]^2 + [z-2a\phi+2b\theta]^2 + [z-2a\phi-2b\theta]^2\right\}$$
$$T = \frac{1}{2}m\left\{[\dot z + a\dot\phi]^2 + [\dot z - a\dot\phi - b\dot\theta]^2 + [\dot z - a\dot\phi + b\dot\theta]^2\right\}$$

(b)
$$\omega^2 = \frac{8k}{m}; \quad [z \quad \theta \quad \phi]^T = [0 \quad 1 \quad 0]^T$$
$$\omega^2 = \frac{1.29k}{m}; \quad \frac{z}{a\phi} = 9.9; \quad \omega^2 = \frac{6.21k}{m}; \quad \frac{z}{a\phi} = -0.42$$

4. (a)
$$V = \frac{1}{2}k\left\{[\theta_2-\theta_1]^2 + [\theta_3-\theta_2]^2 + [\theta_4-\theta_3]^2\right\} + \frac{1}{2}S[\theta_5-\theta_4]^2$$
$$T = \frac{1}{2}J\left\{\dot\theta_1^2 + \dot\theta_2^2 + \dot\theta_3^2 + \dot\theta_4^2 + \dot\theta_5^2\right\}$$

(b) $\dfrac{k}{J}\dfrac{(2\alpha^2 - 2\alpha + 1)}{(1+\alpha^2)}$ (c) $\alpha = -\dfrac{1}{2} \pm \dfrac{\sqrt{5}}{2}; \quad \omega = 0.618\sqrt{\dfrac{k}{J}}, \quad \omega = 1.618\sqrt{\dfrac{k}{J}}$ (d) 0.686%

Final Version