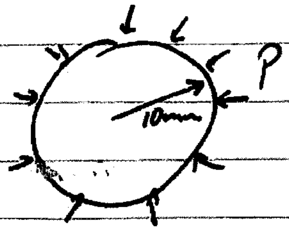
CRIB 3C7 (2011)

Q 1

(a)



Uniform stress $\sigma_{xx} = \sigma_{yy} = -p$

$\Rightarrow$ uniform strain

$$\varepsilon_{xx} = \varepsilon_{yy} = \frac{1}{\bar{E}} \left(\sigma_{xx} - \bar{\nu}\, \sigma_{yy} \right)$$

where $\bar{E} = \dfrac{E}{1-\nu^2}$ $= 231\ \text{GPa}$, $\bar{\nu} = \dfrac{\nu}{1-\nu}$ $= 0.43$ ('plane strain' elastic constants)

$$\frac{\Delta r}{r} = \varepsilon_{xx} = \varepsilon_{yy} = \frac{-p}{\bar{E}} (1 - \bar{\nu})$$ $\cancel{(1-\bar{\nu}) = 1 - \frac{\nu}{1-\nu}}$

4

$$\therefore \Delta r = \frac{-p\, r}{\bar{E}} (1 - \bar{\nu})$$

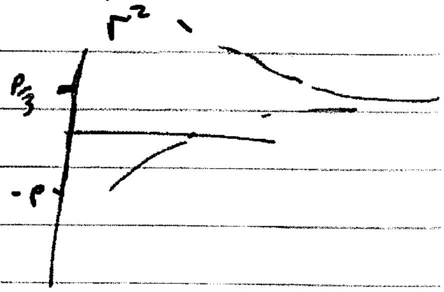
$$= -p \times \frac{10 \times 10^{-3} \times (0.57)}{\cancel{2}\ 231 \times 10^{9}} \quad (\text{in } m, N)$$

$$= -24.7 \times 10^{-15} \times p$$

(sanity check, at yield $p = 400 \times 10^{6}$, $\Delta r \simeq 10 \times 10^{-6}\ m$, $\varepsilon = 1 \times 10^{-3}$ i.e. 0.1%)

21·21

(b) Lame $\sigma_{rr} = A - \frac{B}{r^2}$; $\sigma_{\theta\theta} = A + \frac{B}{r^2}$



at $r = a$ (= 10mm) $\sigma_{rr} = -p$

$r = 2a$ (= 20mm) $\sigma_{rr} = 0$

$$-p = A - \frac{4B}{4a^2}$$

$$0 = A - \frac{B}{4a^2}$$

$$\Rightarrow p = \frac{3B}{4a^2} \qquad B = \frac{4a^2 p}{3}$$

$$A = \frac{p}{3}$$

Find $\varepsilon_{\theta\theta}$ at $r = a$ (where $\sigma_{\theta\theta} = \frac{p}{3} + \frac{4}{3}p = \frac{5}{3}p$)

$$\varepsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu\,\sigma_{rr}\right)$$

$$= \frac{1}{E}\left(\frac{5}{3}p + \nu p\right)$$

$$= \frac{p}{E} \times 1.97$$

$$\Delta r = \varepsilon_{\theta\theta} \times a = \frac{pa}{E} \times 1.97$$

$$= p \times \frac{10 \times 10^{-3}}{210 \times 10^{9}} \times 1.97 \qquad \text{(all N, m)}$$

$$= 93.7 \times 10^{-15}\, p$$

21·31

(c) What pressure causes first yield?

Will occur on inside of outer ring, where

$\sigma_{\theta\theta} > \sigma_{33} = 0 > \sigma_{rr}$

$\therefore$ ~~$\sigma_{\theta\theta}$~~

Assume Tresca (~~[illegible]~~ (Von Mises also OK)

$$\sigma_{\theta\theta} - \sigma_{rr} = \frac{Y}{2} \longleftarrow \text{safety factor of 2}$$

$$\frac{2B}{a^2} = \frac{Y}{2}$$

$$\frac{8}{3} p = \frac{Y}{2}$$

$$p = \frac{3Y}{16}$$

Total misfit δ ~~sum of~~ difference between answers to (a) and (b)

$$\delta = \left[93.7 \times 10^{-15} - (-24.7 \times 10^{-15}) \right] p$$

$$= 118.4 \times 10^{-15} \times \frac{3}{16} \times 400 \times 10^{6}$$

$$= 8.88 \times 10^{-6} \text{ m}.$$

21.37

(d) Difficult to justify either plane stress or plane strain in this case

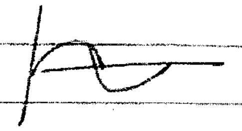
In shaft : long shaft might constrain ε_{33} to be nearly uniform, but correct bc. would be to check axial force $\int \sigma_{33}$ and set this to zero by superposing axial extension (and hence radial contraction due to Poisson effects.

In disk : plane stress reasonable except near interface, where slippage with shaft is probably not possible, so $\sigma_{33} \neq 0$ locally.

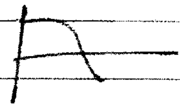
21.41

21.42

2 $$\phi = -\frac{P}{\pi} r\theta\sin\theta$$



$$\frac{\partial\phi}{\partial r} = -\frac{P}{\pi}\theta\sin\theta$$



$$\frac{\partial^2\phi}{\partial r^2} = 0$$

$$\frac{\partial\phi}{\partial\theta} = -\frac{P}{\pi} r\sin\theta - \frac{P}{\pi} r\theta\cos\theta$$

$$\frac{\partial^2\phi}{\partial\theta^2} = -\frac{P}{\pi} r\cos\theta - \frac{P}{\pi} r\cancel{\theta}\cos\theta + \frac{P}{\pi} r\theta\sin\theta$$

$$\sigma_{rr} = \cancel{-\frac{P}{\pi r}\theta\sin\theta} - \frac{2P}{\pi r}\cos\theta + \cancel{\frac{P}{\pi r}\theta\sin\theta}$$

$$= -\frac{2P}{\pi r}\cos\theta$$

$$\sigma_{\theta\theta} = 0$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = 0$$

To give elastic soln, $\nabla^4\phi = 0$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\theta^2}\right)\left(-\frac{2P}{\pi r}\cos\theta\right)$$

$r^{-1} \to -r^{-2} \to 2r^{-3}$

$$= -\frac{4P}{\pi r^3}\cos\theta + \frac{2P\cos\theta}{\pi r^3} + \frac{2P}{\pi r^3}\cos\theta = 0$$

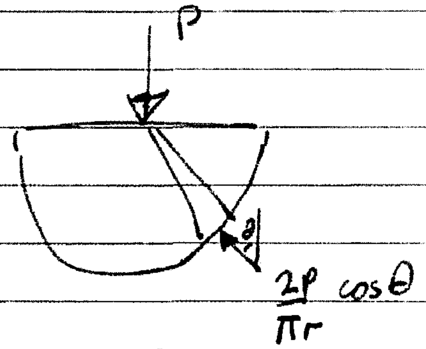
✓

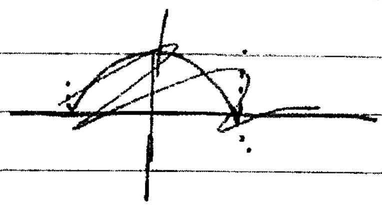
Check B.C.'s at $\theta = \pm \frac{\pi}{2}$, $\sigma_{rr} = \sigma_{r\theta} = 0$

(free surface)

Satisfied ✓

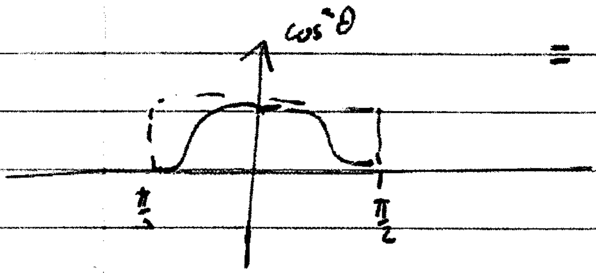
Check equil with applied load. (unit depth into page)





Equil ↕

$$P = \int_{-\pi/2}^{\pi/2} \frac{2P}{\pi \cancel{r}} \cos\theta \cdot \cancel{r\,d\theta} \cdot \cos\theta \; \cancel{r}\, d\theta$$



$$= \frac{2P}{\pi} \cdot \frac{\pi}{2}$$ ✓

Equil ↔ satisfied by symmetry.

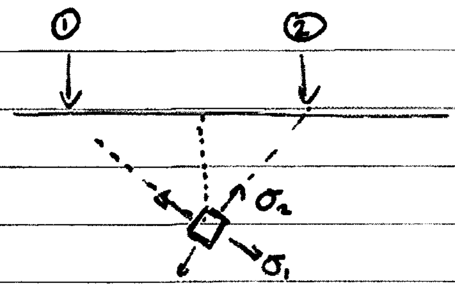
21.55 Thus elastic solution

$\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

(bi) for $y = d$

$\theta_1 = \frac{\pi}{4}$

$\theta_2 = -\frac{\pi}{4}$



$$\sigma_1 = \frac{2P}{\pi\sqrt{2}d} \cdot \frac{1}{\sqrt{2}} = \frac{P}{\pi d}$$

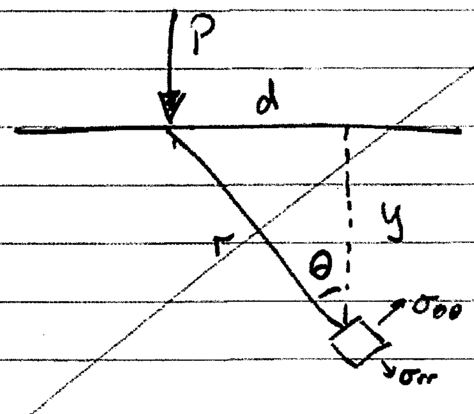
$$\sigma_2 = \frac{P}{\pi d}$$

$$\sigma_1 = \sigma_2, \quad \sigma_d = 0$$

21.57

· b(ii) To do superposition more generally, refer stresses to ~~σ_{xx}, σ_{yy}~~ x, y axes.

For ①



$\frac{d}{r} = \sin\theta$

$\frac{1}{r} = \frac{\sin\theta}{d}$

$\cos\theta = \frac{y}{(d^2+y^2)^{\frac{1}{2}}}$

$$\sigma_{rr} = - \frac{2P}{\pi} \cdot \frac{1}{(d^2+y^2)^{\frac{1}{2}}} \cdot \frac{y}{(d^2+y^2)^{\frac{1}{2}}}$$

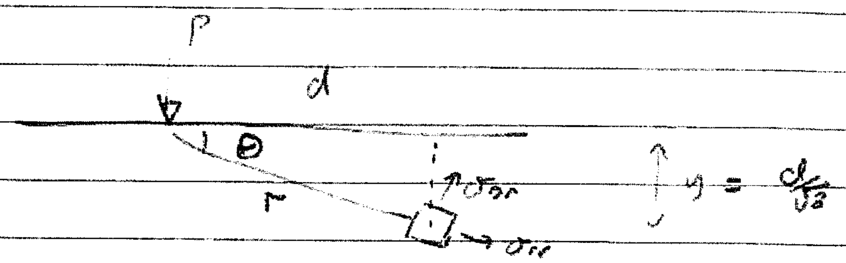
$$= - \frac{2Py}{\pi(d^2+y^2)}$$

11.42

~~Polar~~

b(ii) To do superposition more generally, refer stresses to x, y axes.

For (1)



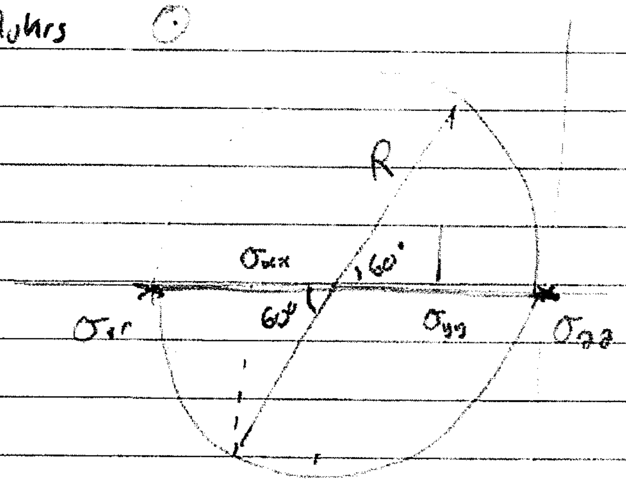
$\theta = 30^\circ \qquad \cos\theta = \frac{\sqrt{3}}{2}$

$r = 2d/\sqrt{3}$

$$\sigma_{rr} = -\frac{2P\sqrt{3}}{\pi \cdot 2d} \cdot \frac{\sqrt{3}}{2} = \frac{3P}{2\pi d}$$

$$\sigma_{\theta\theta} = 0$$

Draw Mohrs ⊙

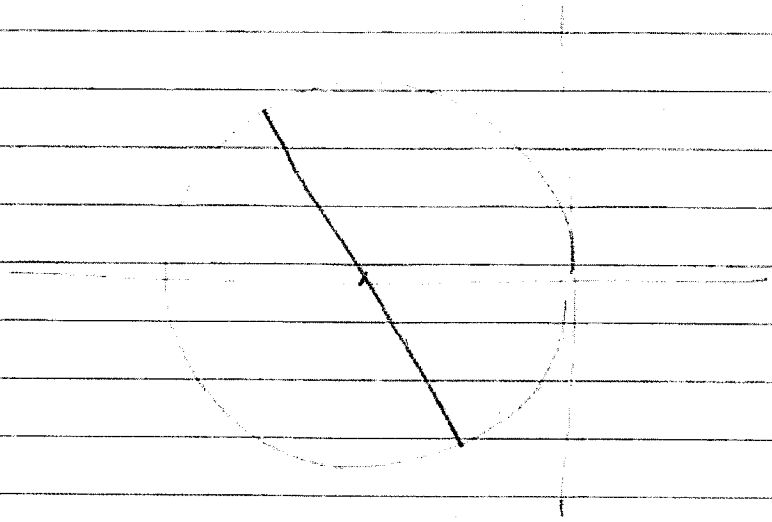


$$R = \frac{3P}{4\pi d}$$

$$\sigma_{xx} = -\frac{3P}{4\pi d} - \frac{3P}{8\pi d} \quad ; \quad \sigma_{yy} = -3P$$

$$= -\frac{9P}{8\pi d} \quad ; \quad \sigma_{yy} = -\frac{3P}{8\pi d}$$

For 2



Same, but σ_{xy} has opposite signs

Superposition : σ_{xy} cancels out (clear from symmetry)

& σ_{xx}, σ_{yy} are principal stresses.

$$\sigma_{xx} = \sigma_2 = -\frac{9}{4}\frac{P}{\pi d} \quad ; \quad \sigma_{yy} = \sigma_1 = -\frac{3}{4}\frac{P}{\pi d}$$

$$\boxed{\therefore \quad \sigma_1 - \sigma_2 = \frac{6}{4}\frac{P}{\pi d} \qquad \left(= \frac{6}{16}\frac{4P}{\pi d}\right)}$$

3

a(i) Need to show that $T = 2\iint_A \psi \, dA$

(ii) $\tau_x = \frac{\partial \psi}{\partial y}$

$= -C\left(y + \frac{3xy}{a}\right)$ where $C = \frac{15\sqrt{3}T}{a^4}$

$\tau_y = -\frac{\partial \psi}{\partial x}$

$= C\left[x - \frac{3x^2}{a} + \frac{3y^2}{2a}\right]$

B.C. is that shear stress $\parallel^{\ell}$ to edge $\Rightarrow$ ψ = constant along edges

$x = -\frac{a}{3}$

$\psi = -C\left(\frac{a^2}{18} + \frac{y^2}{2} + \frac{a^2}{54} - \frac{y^2}{2} - \frac{2a^2}{27}\right) = 0$ ✓

$y = \pm\frac{1}{\sqrt{3}}\left(x - \frac{2a}{3}\right)$

$\psi = -C\left[\left(\cancel{\frac{2x^2}{3}} + \frac{2a^2}{27} - \cancel{\frac{2ax}{9}}\right) - \left(-\cancel{\frac{2ax}{9}} + \cancel{\frac{2x^2}{3}}\right) - \frac{2a^2}{27}\right]$

$= 0$ ✓

(iii) $\nabla^2 \varphi = -2G\beta$ where β is twist / unit length

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}$$

$$= -\left[1 - \frac{3x}{a}\right] - \left[1 - \frac{3x}{a}\right] = -2C$$

$$-2C = 2G\beta$$

$$\beta = \frac{C}{G} = \frac{15\sqrt{3}T}{a^4 G}$$

(b) (i) Torsional stiffness of $\triangleright = \frac{Ga^4}{15\sqrt{3}}$

Area of $\triangleright = \frac{a^2}{\sqrt{3}}$

Circular C.S of radius $b \Rightarrow \pi b^2 = \frac{a^2}{\sqrt{3}}$

$$b = \frac{a}{\sqrt{\pi\sqrt{3}}} \sim \frac{a}{2.33}$$

Torsional stiffness of $\bigcirc = GJ = G\frac{\pi b^4}{2}$

$$= \frac{Ga^4}{6\pi}$$

Relative stiffness $\frac{\bigcirc}{\triangle} = \frac{1}{6\pi} \times 15\sqrt{3}$

$$= 1.37$$

(ii) For $\triangle$, peak shear stress at $x = -\frac{a}{3}, y = 0$

$$\tau_{max} = \frac{15\sqrt{3}\,T}{a^4}\left[-\frac{a}{3} - \frac{3}{2a}\,\frac{a^2}{9}\right]$$

$$= 12.99\,\frac{T}{a^3}$$

For O, $\tau_{max} = \frac{Tb}{J}$

$$= \frac{2T}{\pi b^3}$$

$$b^3 = 0.0788\,a^3$$

$$\tau = \frac{8.08\,T}{a^3}$$

Relative peak stress $\quad O : \triangle = 8.08 : 12.99$

$$= \underline{1 : 1.61}$$

4
(a)

(i) $$\varepsilon_{33} = 0 = \frac{1}{E}\sigma_{33} - \frac{1}{2E}(\sigma_{11} + \sigma_{22})$$

$$\Rightarrow \sigma_{33} = \frac{1}{2}(\sigma_{11} + \sigma_{22})$$

Hence, principal stresses are

$$\sigma_{I} = \frac{\sigma_{11} + \sigma_{22}}{2} + \sqrt{\frac{(\sigma_{11} - \sigma_{22})^2}{4} + \sigma_{12}^2}$$

$$\sigma_{II} = \frac{\sigma_{11} + \sigma_{22}}{2}$$

$$\sigma_{III} = \frac{\sigma_{11} + \sigma_{22}}{2} - \sqrt{\frac{(\sigma_{11} - \sigma_{22})^2}{4} + \sigma_{12}^2}$$

The von-Mises condition is

$$\sqrt{\frac{1}{2}\left((\sigma_{I} - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_{I})^2\right)} = \sigma_{Y}$$

$$\Rightarrow (\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2 = \frac{4}{3}\sigma_{Y}^2$$

(ii) $\sigma_{11} = S, \quad \sigma_{22} = \sigma_{12} = 0$

For Tresca

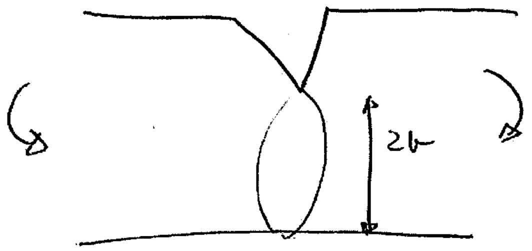
$\sigma_{I} = S, \quad \sigma_{II} = \frac{S}{2}, \quad \sigma_{III} = 0$

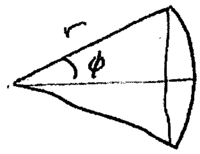
$$\tau_{max} = \frac{\sigma_{I} - \sigma_{III}}{2} = \frac{S}{2}$$

$$\tau_{max} = \frac{S}{2} = \tau_{T} = \frac{\sigma_{T}}{2}$$

$$\frac{S}{\sigma_{T}} = 1$$

4(b)





$r \sin\phi = b$

area of sliding $= 2r\phi$

$= \dfrac{2b}{\sin\phi}\,\phi$

sliding distance $= r\theta = \dfrac{b\theta}{\sin\phi}$

$\Rightarrow k \,\dfrac{2b\phi}{\sin\phi}\,\dfrac{b\theta}{\sin\phi} = M\theta$

$M = \dfrac{k\,2b^2\phi}{\sin^2\phi}$

$M_{min} = 1.38 \times 2kb^2$

$= 2.76\, b^2 k$

Answers to 3C7: Mechanics of Solids (2010-2011)

1. (a) $-24.7\times10^{-15}p\ \ (m)$; where p is in Pa

   (b) $93.7\times10^{-15}p\ \ (m)$; where p is in Pa

   (c) $8.88\times10^{-6}\,m$

2. (b)(ii) $|\sigma_1-\sigma_2|=\dfrac{3P}{2\pi d}$

3. (a)(iii) $\dfrac{15\sqrt{3}T}{a^4G}$

   (b)(i) 1.37
   (b)(ii) 1.61

4. (a)(ii) $S/\sigma_Y=1$

   (b) $2.76b^2k$