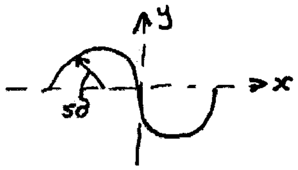
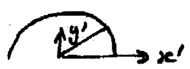
Q1: 

Consider half. 

Area $= \pi R t$ $\quad x' = R\cos\theta$, $\quad y' = R\sin\theta$

$$I_{x'x'} = \int_0^{\pi} t\, y'^2 R\, d\theta = \int_0^{\pi} t R^3 \sin^2\theta\, d\theta = \underline{\pi R^3 t/2}.$$

$$I_{y'y'} = \int_0^{\pi} t\, x'^2 R\, d\theta = \int_0^{\pi} t R^3 \cos^2\theta\, d\theta = \pi R^3 t/2 \quad (\text{again} \checkmark)$$

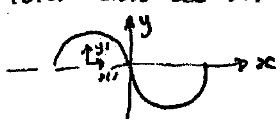
$I_{x'y'} = 0$ by symmetry.

find centroid: $x'_c = 0$

$$(\pi R t) y'_c = \int_0^{\pi} t\, y' R\, d\theta = \int_0^{\pi} t \sin\theta\, R^2 d\theta$$

$$= tR^2 \left[-\cos\theta\right]_0^{\pi} = 2R^2 t$$

$$\rightarrow y'_c = \frac{2R^2 t}{\pi R t} = \underline{\frac{2}{\pi} R}$$ 3

Total cross-section: 

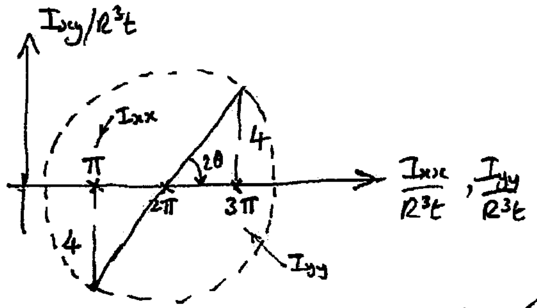
$$I_{xx} = 2 I_{x'x'} = \underline{\pi R^3 t}$$

$$I_{yy} = 2 I_{y'y'} + 2 A \underbrace{x_c^2}_{= R^2} = \pi R^3 t + 2(\pi R t) R^2$$

$$= \underline{3\pi R^3 t}$$

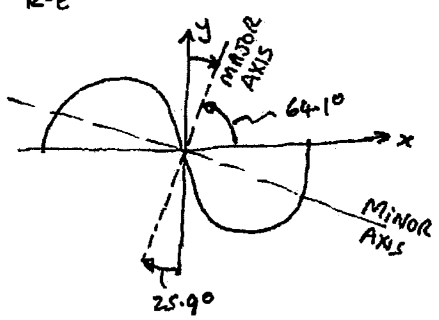
$$I_{xy} = 2 I_{x'y'} + 2A(x_c)(y_c)$$

$$= 0 + 2(\pi R t)(-R)\left(\frac{2}{\pi} R\right) = \underline{-4R^3 t}$$ 2

Mohr's circle 

$\tan 2\theta = \frac{4}{\pi}$ $\quad 2\theta = \tan^{-1}\left(\frac{4}{\pi}\right)$

$\rightarrow \underline{\theta = 25.9°}$



Angle between Major Axis and x-axis
$= 90 - 25.9 = 64.1°$

Q1 cont'd. a) Principal 2^{nd} moments of area:

From Mohr's Circle Geometry

$$\frac{\lambda_{1,2}}{R^3 t} = \underbrace{2\pi}_{\text{centre of Mohr's Circ.}} \pm \underbrace{\sqrt{4^2+\pi^2}}_{\text{radius of Mohr's circ}} = \begin{cases} 11.369 & \text{Major} \\ 1.197 & \text{Minor} \end{cases}$$

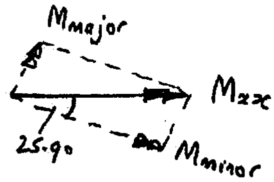
(Alternatively, find eigenvals of $\underline{I} = R^3 t \begin{bmatrix} \pi & 4 \\ 4 & 3\pi \end{bmatrix}$)

Q1 b) Centroid is at $x=0, y=0$ by symmetry
and Shear centre is at $x=0, y=0$ by symmetry.

Q1 c) $\text{BM}\ (M_{xx}) = (wL)\frac{L}{2}$

$$= \rho g (2\pi R t)\frac{L^2}{2} = \underline{\underline{\pi \rho g R t L^2}}$$

Resolve into Princ. Directions.



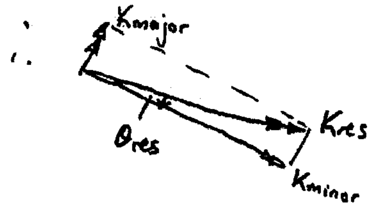
$M_{major} = M_{xx} \sin 25.9° = 0.4368\, M_{xx}$

$M_{minor} = M_{xx} \cos 25.9° = 0.8996\, M_{xx}$

$$K_{major} = \frac{M_{major}}{EI_{major}} = \frac{1}{E}\frac{M_{xx}}{R^3 t}\left(\frac{0.4368}{11.369}\right) = 0.0384\,\frac{M_{xx}}{ER^3 t}$$

$$K_{minor} = \frac{M_{minor}}{EI_{minor}} = \frac{1}{E}\frac{M_{xx}}{R^3 t}\left(\frac{0.8996}{1.197}\right) = 0.7515\,\frac{M_{xx}}{ER^3 t}$$

$\therefore$

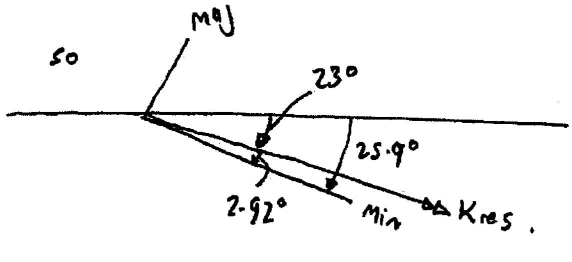


$$|K_{res}| = \frac{M_{xx}}{ER^3 t}\sqrt{0.0384^2 + 0.7515^2} = 0.7525\,\frac{M_{xx}}{ER^3 t}$$

$$\tan\theta_{res} = \frac{K_{maj}}{K_{min}} = \frac{0.0384}{0.7515} = 0.0511$$

$$\Rightarrow \theta_{res} = 2.92°$$

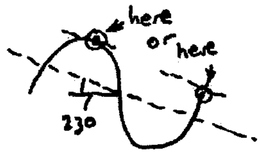
so



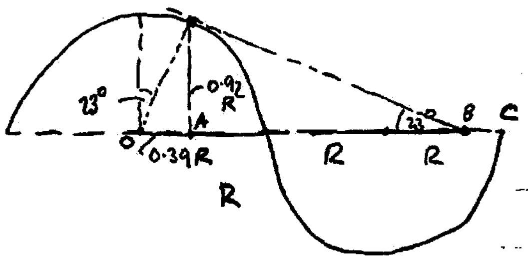
Q1 c) cont'd.

Need to find point on section furthest from axis of curvature K_{res}.

By inspection



Geom:



$$AB = \frac{0.92}{\tan 23°} = 2.167$$

$$OB = 0.39 + 2.167 = 2.55R$$

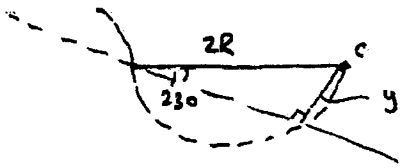
but

$$OC = 3R$$

$\therefore$ B to left of C

$\therefore$ C is worst case

Need distance of C from K_{res} axis:



$$\frac{y}{2R} = \sin 23°$$

$$\rightarrow y = 2R \sin 23°$$

$$= \underline{\underline{0.7815R}}$$

Now $K_{res} = 0.7525 \dfrac{M_{xx}}{ER^3 t} = 0.7525 \left(\dfrac{\pi \rho g L^2 R t}{E R^3 t} \right)$

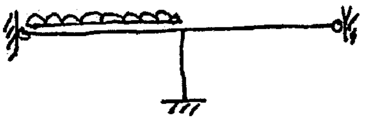
so $\underset{at\,C}{\sigma} = E K_{res} y = E \left(\dfrac{0.7525 \pi \rho g L^2 R t}{E R^3 t} \right) 0.7815 R$

$$= \pi (0.7525)(0.7815) \frac{\rho g L^2}{R}$$

so $\underset{(at\,c)}{\sigma} = \underline{\underline{1.8474 \dfrac{\rho g L^2}{R}}} = \underline{\underline{18.1 \dfrac{\rho L^2}{R}}}$

3D4, 2010-11

Q2.

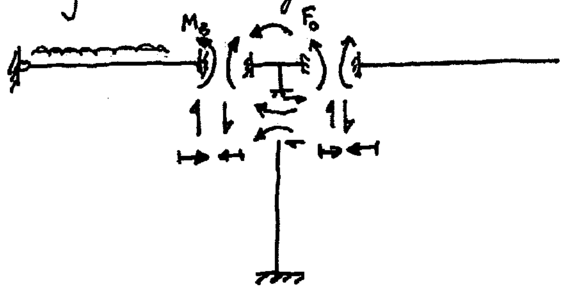
a) 

Kinematical Indeterminacy = 1 

Statical Indeterminacy = 4 

4

b) Kinematically determinate system



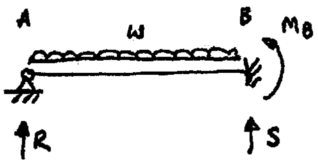
$EA = \infty$

$EI = 5 \times 10^4 \, kNm^2$

$w = 10 \, kN/m$

$M_B = ?$

Real system



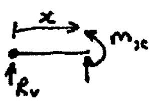
Moments about A ↻+ $\quad M_B + SL = (wL)\frac{L}{2}$

$$S = \frac{wL}{2} - \frac{M_B}{L}$$

Σ vertical forces $\quad R + S = wL$

$$\Rightarrow \quad R = \frac{wL}{2} + \frac{M_B}{L}$$

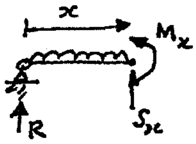
Virtual system



Top: Moments about B: $\quad R_v L = 1 \rightarrow R_v = \frac{1}{L}$

Bot. Moments about RH $\quad m_x = R_v x$

$$\therefore \quad m_x = \frac{x}{L}$$



Moments about RH end: ↻+

$$M_x + wx\left(\frac{x}{2}\right) = Rx$$

$$= \left(\frac{wL}{2} + \frac{M_B}{L}\right)x$$

$$M_x = \left(\frac{wL}{2} + \frac{M_B}{L}\right)x - \frac{wx^2}{2}$$

Real.Virt:

$$\underbrace{M.\theta}_{EXT} = \underbrace{\int_0^L \frac{M_x m_x}{EI}\, dx}_{INT}$$

VIRT

Real

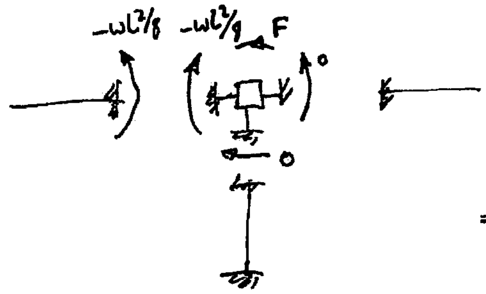
$$1.0 = \frac{1}{EI}\int_0^L \left(\frac{x}{L}\right)\left[\left(\frac{wL}{2} + \frac{M_B}{L}\right)x - \frac{wx^2}{2}\right] dx$$

$$\therefore \quad 0 = \cancel{\frac{w}{2}} \left(\frac{w}{2} + \frac{M_B}{L^2}\right)\int_0^L x^2\, dx - \frac{w}{2L}\int_0^L x^3\, dx$$

$$= \left(\frac{w}{2} + \frac{M_B}{L^2}\right)\frac{L^3}{3} - \frac{w}{2L}\frac{L^4}{4} \quad \Rightarrow \quad \frac{wL^3}{6} + \frac{M_B L}{3} = \frac{wL^3}{8}$$

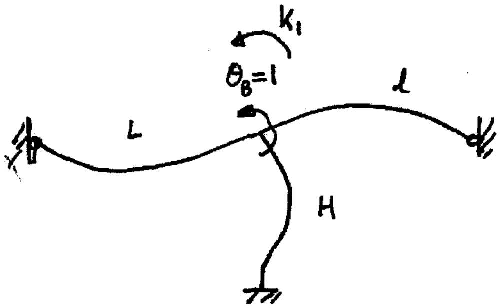
$$\Rightarrow \quad M_B = \frac{3}{L}\left[\frac{wL^3}{8} - \frac{wL^3}{6}\right] = 3wL^2\left[\frac{6-8}{48}\right] = -3wL^2\,\frac{2}{48}$$

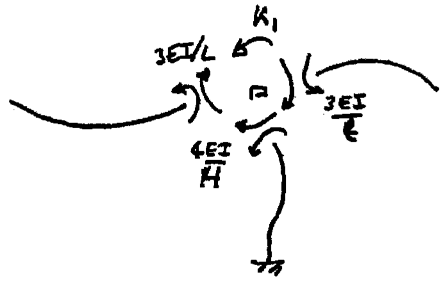
$$= -\frac{wL^2}{8}$$



Joints fixed.

$$\Rightarrow \quad F = -\frac{wL^2}{8}$$





$$K_1 = \frac{3EI}{L} + \frac{3EI}{\ell} + \frac{4EI}{H}$$

$$F + K_1\theta_B = 0 \quad \Rightarrow \quad -\frac{wL^2}{8} + \left(\frac{3EI}{L} + \frac{3EI}{\ell} + \frac{4EI}{H}\right)\theta_B = 0$$

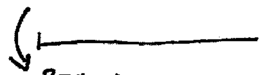
$$\Rightarrow \quad \theta_B = \frac{wL^2/8}{\left(\frac{3EI}{L} + \frac{3EI}{\ell} + \frac{4EI}{H}\right)} = \frac{10(6)^2/8}{\left(\frac{3}{6} + \frac{3}{8} + \frac{4}{4}\right)EI} = \frac{45}{(1.875)\,EI}$$

$$= \frac{24}{EI} = \frac{24}{5\times10^4} = \underline{4.8\times10^{-4}\ \text{rads.}}$$

On BA: $\quad -wL^2/8 \; + \; \frac{3EI}{L}\theta_B \quad = -\frac{wL^2}{8} + \frac{3EI}{L}\left(\frac{24}{EI}\right)$

$$= -\frac{10(6)^2}{8} + \frac{72}{6} = -45 + 12 = \underline{-33}\ \text{kNm}$$

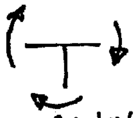
On BC :



$$\frac{3EI}{L}\theta_B = \frac{3EI}{8}\,\frac{24}{EI} = 9 \text{ kNm}$$

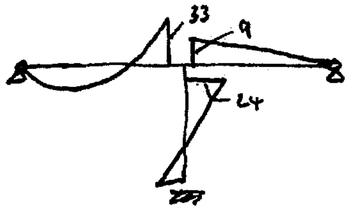
On BD:

$$\frac{4EI}{H}\theta_B = \frac{4EI}{4}\,\frac{24}{EI} = 24 \text{ kNm}$$



in equilibrium.

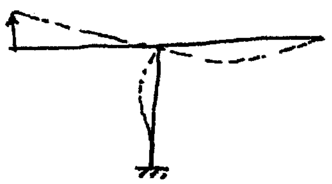
BMD

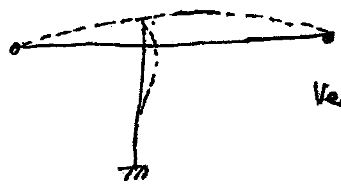


11

Q2 c).

Vertical reaction at A





Vertical reaction at D.

5

3D4 2010-2011

Q3 (a) j^{th} column = vector of nodal forces required to give unit displacement in j^{th} component of nodal displacement vectors, and zero displacement along all other components. 2

(b) Marks awarded for clear explanation of ~~eigenvalues~~ eigenvectors as static modes (e.g. force of one "shape" gives displacement of same "shape") and of eigenvalues as the generalised stiffnesses of these deflected shapes. (More general than concept of "buckling mode" which is only applicable at point of buckling). Useful in stability analysis as the eigenvalues (and the positivity thereof) measures how stable the structure is, and buckling corresponds to the first time an eigenvalue passes through zero. 4

c) i) By inspection, stiffness matrix

$$= \frac{EI}{L}\begin{bmatrix} s+7 & sc \\ sc & s+6 \end{bmatrix}$$

This buckles when $\det = 0 \Rightarrow (s+6)(s+7) - s^2c^2 = 0$

$$s^2 + 13s + 42 - s^2c^2 = 0$$

$$s^2(1-c^2) + 13s + 42 = 0.$$

Interpolate:

P/P_E	s	c	$s^2(1-c^2)$	$13s+42$	
2.0	0.1428	24.6841	−12.40	43.48	
2.2	−0.5194	−7.5107	−14.9485	35.24	
2.4	−1.3006	−3.3703	−17.5	25.092	← cross here.
2.6	−2.2490	−2.2312	−20.122	12.76	$\sim P/P_E = 2.5$

$P/P_E \approx 2.5 \qquad P/P_E = \frac{\pi^2 EI}{L_E^2}\frac{L^2}{\pi^2 EI} \rightarrow \frac{L_E}{L} = \sqrt{P_E/P} = \sqrt{\frac{1}{2.5}}$

$= \underline{0.6325}$ 6

3D4 2010-2011

Q3 cont'd.

c ii) $\begin{bmatrix} M_B \\ M_E \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} s+7 & sc \\ sc & s+6 \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_E \end{bmatrix}$

and $P = P_E$, so $s = 2.4674$, $c = 1.0$

$$\therefore \begin{bmatrix} M_B \\ M_E \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} 9.4674 & 2.4674 \\ 2.4674 & 8.4674 \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_E \end{bmatrix}$$

$$\underline{M} = \begin{bmatrix} PL/100 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\pi^2 EI}{L^2}\frac{L}{100} \\ 0 \end{bmatrix} = \frac{EI}{L}\begin{bmatrix} \pi^2/100 \\ 0 \end{bmatrix}$$

($\frac{EI}{L}$'s cancel)

$$\rightarrow \begin{bmatrix} \theta_B \\ \theta_E \end{bmatrix} = \begin{bmatrix} 9.4674 & 2.4674 \\ 2.4674 & 8.4674 \end{bmatrix}^{-1} \begin{bmatrix} \pi^2/100 \\ 0 \end{bmatrix}$$

$$= \frac{1}{(8.4674)(9.4674) - (2.4674)^2} \begin{bmatrix} 8.4674 & -2.4674 \\ -2.4674 & 9.4674 \end{bmatrix} \begin{bmatrix} \pi^2/100 \\ 0 \end{bmatrix}$$

denom = 74.076

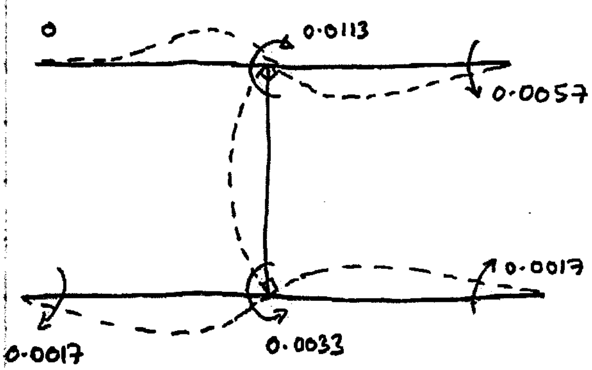
$$\rightarrow \theta_B = \frac{8.4674\ \pi^2/100}{74.076} = \underline{0.0113 \text{ rads}}$$

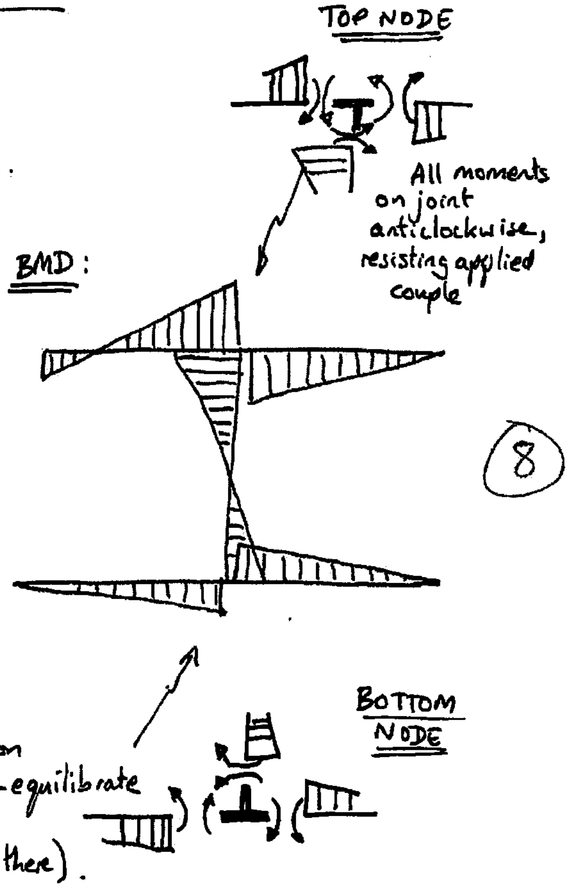
$$\theta_E = \frac{-2.4674\ \pi^2/100}{74.076} = \underline{-0.0033 \text{ rads}}$$

Also $|\theta_c| = |\theta_B|/2 = 0.0057$ rads

$|\theta_D| = |\theta_F| = |\theta_E|/2 = 0.0017$ rads.

Deflected shape:

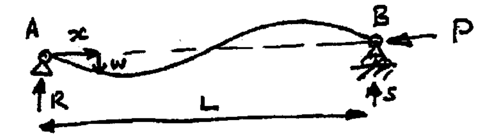




8

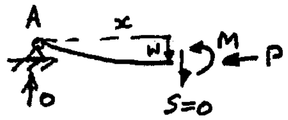
3D4 2010-11

Q4. i)



M_A ⟲: $S = 0$

M_B ⟲: $R = 0$



Equilib: M_A ⟲ $-M + Pw = 0$

Const.t/Compat: $M = -EI\left(\frac{d^2w}{dx^2} - \frac{d^2w_0}{dx^2}\right)$

$\Rightarrow \quad EI w'' + Pw = EI w_0''$ $\qquad w_0$ = init. imperfection

Let $w_0 = b \sin\left(\frac{\pi x}{L}\right)$
Initial half-sine wave bow
and $w = a \sin\left(\frac{\pi x}{L}\right)$
generally

$\left(-\frac{\pi^2 EI}{L^2} + P\right) a \sin\frac{\pi x}{L} = \left(-\frac{\pi^2 EI}{L^2}\right) b \sin\frac{\pi x}{L}$

write $P_E = \pi^2 EI/L^2$

$\Rightarrow \quad a = \frac{b P_E}{P_E - P} = \frac{b}{1 - P/P_E}$

Moment at centre of beam $\quad M_{centre} = Pa = Pb/(1 - P/P_E)$

Extreme fibre stress on concave side $\quad \sigma_{max} = \frac{P}{A} + \frac{M y_{max}}{I} = \frac{P}{A} + \frac{Pb}{(1-P/P_E)} \frac{y_{max}}{A r^2}$

so $\quad \sigma_{max} = \sigma + \frac{\sigma}{(1 - \sigma/\sigma_E)}\left(\frac{b y_{max}}{r^2}\right)$ $\equiv \eta$ imperfection parameter

Setting $\sigma_{max} = \sigma_y$ yield stress $\Rightarrow$

$$\sigma_y = \sigma + \eta \frac{\sigma}{(1 - \sigma/\sigma_E)} = \sigma + \frac{\eta \sigma \sigma_E}{(\sigma_E - \sigma)}$$

whence $\quad (\sigma_y - \sigma)(\sigma_E - \sigma) = \eta \sigma \sigma_E$

8

3D4 2010-2011

Q4

b) i) Marks awarded for clear explanation of additional stiffness associated with warping restraint (2)

ii) 406×140×46 UB.

$L = 6m$, $\quad I_{minor} = 538 \times 10^{-8}\ m^4$

$\quad J = 19 \times 10^{-8}\ m^4$

$$G = \frac{E}{2(1+\nu)} = \frac{E}{2.6} \quad \text{for } \nu = 0.3 \text{ (steel) and } E = 210 \times 10^9\ N/m^2.$$

$$\Gamma = \frac{ID^2}{4}, \quad D = 403.2 - 11.2 = 392\ mm = 0.392\ m$$

$$\Gamma = \frac{538 \times 10^{-8} (0.392)^2}{4} = \underline{20.7 \times 10^{-8}}\ m^6.$$

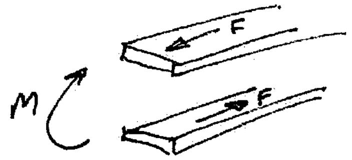
$$\sqrt{1 + \frac{\pi^2}{L^2}\frac{E\Gamma}{GJ}} = \sqrt{1 + \frac{\pi^2}{6^2}(2.6)\frac{20.7}{19}} = \sqrt{1.78} = \underline{1.33}$$

$$\frac{\pi}{L}\sqrt{GJEI} = \frac{\pi}{6}(210 \times 10^9)\sqrt{\frac{(19)(538)}{2.6}} \times 10^{-8}$$

$$= \underline{68.9\ kNm}$$

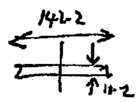
so $M_{cr} = 1.33(68.9) = \underline{91.7\ kNm}$ (6)

iii)



$FD = M \qquad D$ = dist between flange centroids $= 0.392m$

Also $F_{cr} = \frac{\pi^2 EI}{L^2}$ $\qquad$ $I = \frac{bd^3}{12}$ for flange alone



$$= \frac{(0.0112)(0.1422)^3}{12} = \underline{2.684 \times 10^{-6}\ m^4}$$

$$F_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2}{6^2}(210 \times 10^9)(2.684 \times 10^{-6})\ N = 155\ kN$$

$$M_{cr} = FD = 155\ kN \times 0.392\ m = \underline{60.8\ kNm}$$

Q4 a) Cont'd.

This should equal the second term (the warping restraint term) in the above eqn.

(Check: $M_{2nd} = \frac{\pi}{L}\sqrt{GJEI}\left(\frac{\pi^2}{L^2}\frac{E\Gamma}{GJ}\right)^{1/2}$ - - -

$= 68.9 \text{ kNm} \times \sqrt{0.78} = \underline{60.8 \text{ kNm}}$ ✓)

So discrepancy is omission of first term which arises due to St Venant torsional resistance, (due to shear stresses ).

(4)

3D4 Structural Analysis and Stability 2010-2011
Answers

1. a) $11.369R^3t$, $1.197R^3t$, 64.1°
   c) $1.8474\rho gL^2/R$

2. a) Kinematical = 1, Statical = 4 (Other answers possible if well-argued).
   b) BA -33kNm, BC 9kNm, BD 24kNm

3. c) i) $0.6325L$
   ii) 0.0113, -0.0033

4. b) ii) 91.7kNm
   iii) 60.8kNm