First eqn. — multiply by weight function and integrate

Q1. a) $$-\int_L v \frac{d\sigma}{dx}\, dx = \int_L v f\, dx$$

Integrate by parts,

$$\int_L \frac{dv}{dx} \sigma\, dx = \int_L v f\, dx + v\sigma \Big|_{x_0}^{x_1}$$

Insert constitutive model,

$$\rightarrow \int \frac{dv}{dx} E\left(\frac{du}{dx} - \beta(T - T_0)\right) dx = \int_L v f\, dx + \underset{\uparrow}{v\sigma} \Big|_{x_0}^{x_1}$$

insert traction boundary condition

Second eqn. (multiply by weight function w & integrate)

$$\int_L w \dot{T}\, dx - \int_L w k \frac{d^2 T}{dx^2}\, dx + \int_L w E\beta T_0 \frac{\partial^2 u}{\partial x \partial t}\, dx = 0$$

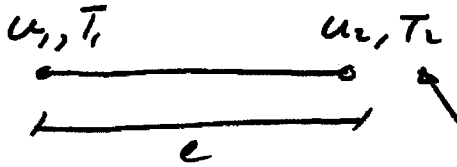
Apply integration by parts,

$$\int_L w \dot{T}\, dx + \int_L \frac{dw}{dx} k \frac{dT}{dx}\, dx - w k \frac{\partial T}{\partial x}\Big|_{x_0}^{x_1} + \int_L w E\beta T_0 \underbrace{\frac{\partial \dot{u}}{\partial x}}\, dx = 0$$

$= \partial^2 u / \partial x \partial t$

Do not integrate by parts since this would lead to an unwelcome boundary term.

Q1 b)



linear element

two dofs per node (disp & temp)

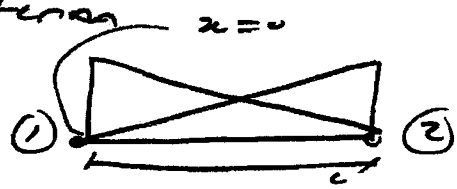
'mass' matrix

$$\underline{\underline{M}}\,\dot{\underline{a}} + \underline{\underline{K}}\,\underline{a} = \underline{f}$$

$$\underline{\underline{M}} = \left[\begin{array}{c|c} \underline{0} & \underline{0} \\ \hline \int E\beta T_0 \underline{\underline{N}}^T \underline{\underline{B}}\,dx & \int_{x_0}^{x_1} \underline{\underline{N}}^T \underline{\underline{N}}\,dx \end{array}\right]$$

$\int w\,\dot{T}\,dx$ term

$\int w\,E\beta T_0 \dfrac{\partial \dot{u}}{\partial x}\,dx$ term



$N_1 = x/l$

$N_2 = -x/l + 1$

$dN_1/dx = 1/l$

$dN_2/dx = -1/l$

$$E\beta T_0 \int \underline{\underline{N}}^T \underline{\underline{B}}\,dx = \int_0^l \begin{bmatrix} x/l \\ -x/l+1 \end{bmatrix} \begin{bmatrix} 1/l & 1/l \end{bmatrix} dx$$

$$= / E\beta T_0 \int_0^l \begin{bmatrix} x/l^2 & -x/l^2 \\ -\dfrac{x}{l^2} + 1/l & x/l^2 - 1/l \end{bmatrix} dx$$

$$= / E\beta T_0 \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

Q1 b) cont $\int_0^l \underline{N}^T \underline{N} \, dx = \begin{bmatrix} l/3 & l/6 \\ l/6 & l/3 \end{bmatrix}$

$$\Rightarrow \underline{\underline{M}}_c = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha/2 & -\alpha/2 & l/3 & l/6 \\ \alpha/2 & -\alpha/2 & l/6 & l/3 \end{bmatrix}$$

$$\alpha = E\beta T_0$$

c) Backward Euler:

$$y_{n+1} = y_n + \Delta t \, \dot{y}_{n+1}$$

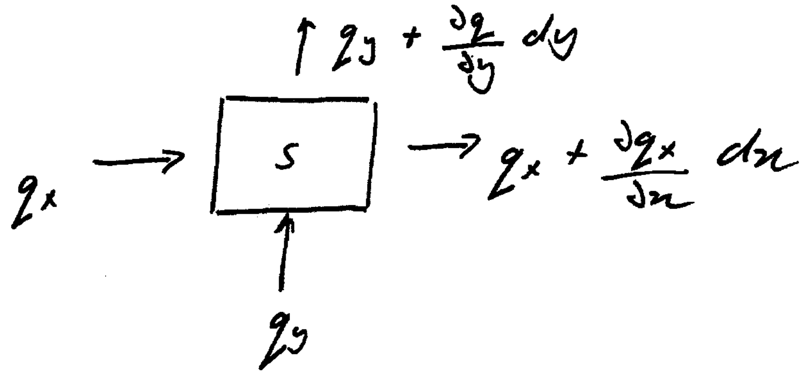
$$\rightarrow \dot{y}_{n+1} = \frac{y_{n+1} - y_n}{\Delta t}$$

Insert into $\underline{\underline{M}} \, \dot{\underline{a}} + \underline{\underline{K}} \, \underline{a} = \underline{b}$

$$\rightarrow \underline{\underline{M}} \, \frac{\underline{a}_{n+1} - \underline{a}_n}{\Delta t} + \underline{\underline{K}} \, \underline{a}_{n+1} = \underline{b}_{n+1}$$

$$\rightarrow \underline{\underline{M}} \, \underline{a}_{n+1} + \Delta t \, \underline{\underline{K}} \, \underline{a}_{n+1} = \Delta t \, \underline{b}_{n+1} + \underline{\underline{M}} \, \underline{a}_n$$

Q2 a) $q_x = -k\dfrac{\partial h}{\partial x}$, $q_y = -k\dfrac{\partial h}{\partial y}$



$$-\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} + s = 0 \quad \text{(conservation of mass)}$$

$$\Rightarrow k\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) + s = 0$$

b) Multiply by weight function,

$$\underbrace{\int_\Omega k\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right)}_{-\nabla\cdot \underline{q} \quad \left(\underline{q} = \left(-k\frac{\partial h}{\partial x}, -k\frac{\partial h}{\partial y}\right)\right)} \cdot w_0 \, d\Omega + \int_\Omega s w_0 \, d\Omega = 0$$

Integrate by parts,

$$\int_\Omega \underline{q}\cdot\nabla w_0 \, d\Omega + \int_{\partial\Omega} -\underline{q}\cdot\underline{n} w_0 \, d\Gamma$$

$$+ \int_\Omega s w_0 \, d\Omega = 0$$

Insert $\underline{q}$ and re-arrange,

$$k \int_{\Omega} \left(\frac{\partial h}{\partial x} \frac{\partial w_0}{\partial x} + \frac{\partial h}{\partial y} \frac{\partial w_0}{\partial y} \right) d\Omega$$

$$= \int_{\Omega} s\, w_0 \, d\Omega - \int_{\partial\Omega} \underline{q} \cdot \underline{n} w_0 \, d\Gamma^{+}$$

→ $w_0 = 0$ where Dirichlet condition is applied (h prescribed)

→ Γ_q^{+} : Neumann boundary where flux is prescribed

$\bar{q}$: $\underline{q} \cdot \underline{n}$

c) $\underline{\underline{K}} = \int_{\Omega_e} \underline{\underline{B}}^T \underline{\underline{D}} \, \underline{\underline{B}} \, d\Omega$ (element)

$$K_{11} = \sum_{ij} \int_{\Omega_e} B_{i1} D_{ij} B_{j1}$$

$$= \int_{\Omega_e} B_{11} D_{11} B_{11} + B_{21} D_{22} B_{21} \, d\Omega$$

$$B_{11} = \frac{1}{2A}(y_2 - y_3)$$

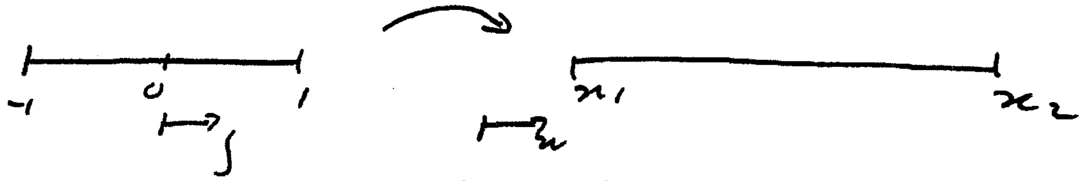
$$B_{21} = \frac{1}{2A}(x_3 - x_2)$$

$A = 18.75$

$$K_{11} = 18.75 \left(k \left(\frac{1}{2(18.75)} (-5) \right)^2 + k \left(\frac{1}{2(18.75)} (-7.5) \right)^2 \right)$$

↑ constant derivative, $= 1.08333k$ $(13/12 \; k)$

Q3 a) Map from reference (parent) domain to real domain using shape functions



$$x(\xi) = \sum N^k(\xi)\, x^k$$

b) - Arbitrary quads only possible with isoparametric mapping
- Curved boundaries
- Straightforward numerical integration
- Shape functions defined on common reference elements

c) $$f_3 = \int_\Omega N_3\, q\, d\Omega \qquad (q = 2000)$$

$$= \int_{-1}^{1}\int_{-1}^{1} N_3(\xi,\eta)\, \underbrace{|J|}_{\text{determinant of Jacobian}}\, d\xi\, d\eta$$

Using numerical integration,

$$f_3 = \sum_i N_3(\xi_i, \eta_i)\, \underbrace{w_i}_{\text{weight}}\, |J_i|\, q \qquad (1)$$

$$\underline{\underline{J}}^e = \frac{1}{4}\begin{bmatrix} \eta-1 & 1-\eta & 1+\eta & -\eta-1 \\ \xi-1 & -\xi-1 & 1+\xi & 1-\xi \end{bmatrix}\begin{bmatrix} 0 & 0.5 \\ 0 & 0 \\ 1 & 0.5 \\ 1 & 0.75 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 0 & 0.5(\eta - 1 + 1 + \eta) - 0.75(\eta + 1) \\ 2 & 0.5(\xi - 1) + 0.5(1+\xi) + 0.75(1-\xi) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.0625\eta - 0.1875 \\ 0.5 & 0.0625\xi + 0.1875 \end{bmatrix}$$

Use 4-point quadrature scheme (Gauss)

GP1: $\xi = -\frac{1}{\sqrt{3}}$, $\eta = -1/\sqrt{3}$

$$J = \begin{bmatrix} 0 & -0.2236 \\ 0.5 & - \end{bmatrix} \rightarrow |J| = 0.112$$

GP2: $\xi = 1/\sqrt{3}$, $\eta = -1/\sqrt{3}$

$|J| = 0.112$

GP3: $\xi = -1/\sqrt{3}$, $\eta = 1/\sqrt{3}$

$|J| = 0.0757$

GP4: $\xi = 1/\sqrt{3}$, $\eta = 1/\sqrt{3}$

$|J| = 0.0757$

Evaluate shape function N_3 at Gauss points

GP1: $N_3 = 0.04466$

GP2: $N_3 = 0.1667$

GP3: $N_3 = 0.1667$

GP4: $N_3 = 0.62201$

Apply numerical integration (see (1)),

$$f_3 = (0.04466 \times 0.112 + 0.1667 \times 0.112$$
$$+ 0.1667 \times 0.0757 + 0.6220 \times 0.0757)q$$
$$= 0.8338\,q = \underline{-166.75}$$

d) Mesh is too coarse. Result will be too stiff.

Q4 a) i) Recall $\dot{y} = f$

Given eqn: $\dot{y} = \underbrace{-\lambda y}_{f}$

Insert into forward Euler expression:

$$y_{n+1} = y_n - \Delta t \lambda y_n$$
$$= \underbrace{(1 - \Delta t \lambda)}_{A\text{: amplification factor.}} y_n$$

Stable when $|A| \leq 1$

$$\Rightarrow (1 - \Delta t \lambda)^2 = 1$$
$$\rightarrow \Delta t \leq 2/\lambda \quad (\text{since } \lambda > 0)$$

ii) $$y_{n+1} = y_n + \Delta t f^*$$

$$y^*_{n+1/2} = y_n - \frac{\Delta t}{2} \lambda y_n$$

$$\rightarrow f^*_{n} = -\lambda y^*_{n+1/2} =$$

Proceed as with forward Euler, but instead of using $f_n = -\lambda y_n$, use $f^*_{n+1/2} = -\lambda y^*_{n+1/2}$,

$$y_{n+1} = y_n - \Delta t \lambda \left(y_n - \frac{\Delta t}{2} \lambda y_n\right)$$
$$= \underbrace{\left(1 - \Delta t \lambda + \frac{\Delta t^2 \lambda^2}{2}\right)}_{A} y_n$$

$$|A| \leq 1 \rightarrow \Delta t \leq 2/\lambda$$

* Can insert $\Delta t = 2/\lambda$ to verify that method is on stability limit.

Q 4 ~~$\exists$~~ a) (ii)

Advantage of scheme is 2nd order accuracy ✓

b) For heat conduction

$$\Delta t_{crit} \sim \frac{1}{\lambda}, \quad \lambda_{max} \sim 1/h^2$$

$$\rightarrow \quad \Delta t_{crit} \sim h^2$$

For elastodynamics

$$\Delta t_{crit} \sim 1/\sqrt{\lambda}$$

$$\rightarrow \quad \Delta t_{crit} \sim h$$

Critical time step reduce more rapidly for heat conduction as mesh is refined.

c) As the polynomial order is increased, so does the maximum eigenvalue $\rightarrow$ critical tim step decreases.