3F1 Signals and Systems 2011
Crib

Question 1

(a) (i)

$$
\begin{aligned}
Y(z) &= \frac{z^{-1}}{(1-z^{-1})(1-3z^{-1})} \\
&= z^{-1}\left[-\frac{1}{2(1-z^{-1})} + \frac{3}{2(1-3z^{-1})}\right]
\end{aligned}
$$

So

$$
y_k = \begin{cases} 0 & k \le 0 \\ -\frac{1}{2} + \frac{3}{2}3^k & k \ge 1 \end{cases}
$$

(ii) No, $Y(z)$ has a pole with magnitude greater than 1.

(b)

$$
U(z) = \frac{1}{1-z^{-1}}
$$

$$
G(z) = Y(z)/U(z) = \frac{z^{-1}}{1-3z^{-1}} = \frac{1}{z-3}
$$

For the difference equation note that $(z-3)Y(z) = U(z)$, so

$$
y_{k+1} - 3y_k = u_k
$$

(c) (i)

$$
T_{r \to y} = \frac{G(z)}{1-\lambda G(z)} = \frac{1/(z-3)}{1-\lambda/(z-3)} = \frac{1}{z-3-\lambda}
$$

(ii) For stability we need

$$
|\lambda + 3| < 1 \Rightarrow -4 < \lambda < -2
$$

(iii)

$$\frac{G(z)}{1-\lambda z^{-1}G(z)} = \frac{z}{z(z-3)-\lambda} = \frac{z}{z^2-3z-\lambda}$$

To find the poles we have

$$z^2 - 3z - \lambda = 0 \Rightarrow z = \frac{3 \pm \sqrt{9+4\lambda}}{2}$$

So for at least one solution $|z| > 1$ hence the system cannot be stabilized.

1 Question 2

(a) (i) The statement is false.

$$s = \frac{z-1}{T} \Rightarrow \mathrm{Re}(s) = -\frac{1}{T} + \frac{1}{T}\mathrm{Re}(z)$$

So for $\mathrm{Re}(s) < 0$ $|z|$ can be arbitrarily large.

(ii) The statement is true.

$$\begin{aligned} s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) &= \frac{2}{T}\left(\frac{(z-1)(z^*+1)}{|z+1|^2}\right) \\ &= \frac{2}{T}\left(\frac{|z|^2+z-z^*-1}{|z+1|^2}\right) \\ &= \frac{2}{T}\left(\frac{|z|^2-1+2\mathrm{Im}(z)}{|z+1|^2}\right) \end{aligned}$$

So

$$\mathrm{Re}(s) = \frac{2}{T}\left(\frac{|z|^2-1}{|z+1|^2}\right)$$

and $\mathrm{Re}(s) < 0 \Rightarrow |z|^2 - 1 < 0 \Rightarrow |z| < 1$

(iii)

$$G(z) = \frac{z-1}{z}\mathcal{Z}\left\{\left[\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)\right]_{t=kT}\right\}$$

(b) See notes section 2.1.1 on pp 17-18, except that sigma is replaced by σ_1 and σ_2 for the x_1 and x_2 directions. Result in (2.6) is modified accordingly.

For the target, let $\sigma_1 = \sigma_2 = \sigma$ to get result of (2.6), then derive result of (2.7). Finally integrate (2.7) to get the radial cdf as

$$F(r) = 1 - \exp(-r^2/2\sigma^2)$$

Then put in $sigma = 15$ and $r = 10, 20$ and 30. Mean score per shot is then

$$F(10) * 100 + (F(20) - F(10)) * 50 + (F(30) - F(20)) * 25.$$

2 Question 3

(a) See sec 4.4 p 43 of notes. All ergodic processes must also be stationary but stationary processes need not be ergodic. Hence stationarity is a necessary condition for ergodicity, but is not sufficient.

(b) PSD is the Fourier transform of the ACF. See equ (4.20) and (4.21) on p 46 of notes.

(c) See sec 4.5.2 on pp 47-48 of the notes. Result is equ (4.27). The quick way to get this, is to observe that the PSD of the output is the PSD of the input multiplied by the power gain of the filter at each frequency.

(d) Take inverse Fourier transform of (4.27) to get equ (4.25).

(e) If the PSD or ACF of X and of Y can both be measured, then the squared magnitude of the frequency response is given by S_Y/S_X. Taking the square root of this gives the magnitude of the frequency response. Note that we cannot obtain the phase of the frequency response this way (to do this we need to calculate the cross spectrum of the output and the input).

3 Question 4

(a) See figure at top of p 13 in the notes. Let $f(x) = LHS - RHS = \ln x - (x-1)$

$$df/dx = (1/x) - 1$$
$$d^2f/dx^2 = (-1/x^2)$$

Hence $f(x)$ reaches a maximum when $x = 1$, since $df/dx = 0$ and the 2nd derivative is negative. At this point $f(x) = 0$.

Hence $f(x) \leq 0$ and the result is proven. Equality is at the maximum, at which $x = 1$.

(b) See summary of proof on p 13 of the notes, and also the full proof in the crib for examples sheet Q1. pmf = 1/N for all N states.

(c) $H(Y)$ requires us to calculate $p_Y(y)$ for $y = A, B, C$.

Now

$$\begin{aligned} p_Y(A) &= P(A|x = A).p_X(A) + P(A|x = B).p_X(B) + P(A|x = C).p_X(C) \\ &= (1-b).1/3 + b.1/3 + 0.1/3 = 1/3 \end{aligned}$$

Similarly

$$p_Y(B) = (b + (1-2b) + b).1/3 = 1/3$$

and

$$p_Y(C) = (0 + b + (1-b)).1/3 = 1/3$$

So

$$H(Y) = -3.1/3.\log_2(1/3) = \log_2(3)$$

and

$$H(Y|X) = H(Y|A).1/3 + H(Y|B).1/3 + H(Y|C).1/3$$

Now

$$H(Y|A) = H(Y|C) = -(1-b).\log_2(1-b) - b.\log_2(b)$$

and

$$H(Y|B) = -2.b.\log_2(b) - (1-2b).\log_2(1-2b)$$

Hence

$$H(Y|X) = -1/3[4b\log_2(b) + 2(1-b)\log_2(1-b) + (1-2b)\log_2(1-2b)]$$

(d) $I(X;Y) = H(Y) - H(Y|X)$ (see p 29 of notes with X and Y swapped.)

$$H(Y) = 1.5850 bit/sym.$$

With $b = 0.1$:

$$\begin{aligned} H(Y|X) &= -1/3[0.4(-3.3219) + 1.8(-0.1520) + 0.8(-0.3219)] \\ &= 1/3.1.8599 = 0.6200 bit/sym \end{aligned}$$

Hence the $I(X;Y) = 1.5850 - 0.6200 = 0.9650$ bit/sym.

The channel capacity is the maximum of the mutual information over all possible sets of input probabilities. This usually happens when H(Y) is maximised, and hence also when H(X) is maximised, which is the case here when all three states are equiprobable.

Hence the channel capacity = 0.9650 bit/sym.

CORRECTION: a mistake was spotted in this crib. The argument above is approximate but in this case somewhat incorrect. Since the channel effectively has one noisy input symbol B and two less noisy input symbols A and C, it is intuitively clear that the capacity-achieving input distribution should assign more weight to A and C and less weight to B. A quick calculation of mutual information with the input distribution $P_X(A) = P_X(C) = 0.4$ and $P_X(B) = 0.2$ gives $I(X;Y) = 0.995458$ bit/symbol, which is superior to the result quoted above. An exact calculation of capacity can be done by first proving that the capacity-achieving distribution must be symmetric in A and C, making it a one-parameter distribution, then expressing mutual information as a function of this parameter and maximising by setting its derivative to zero. This is a long and tedious calculation that we would not expect students to do without hints or help in the context of a tripos examination, and yields a capacity of $C = 0.995474$ bit/sym. **END OF CORRECTION.**

(e) The output entropy $H(Y)$ is the maximum capacity of the channel in the absence of errors (when $H(Y|X) = 0$). The ratio of the channel capacity with $b = 0.1$ to the error-free capacity tells us the maximum ratio of data bits to total bits in an ideal error-correcting code for this channel. Hence the redundancy of the code (ratio of total bits to data bits) needs to be at least

$$1.5850 : 0.9650 = 1.6425 : 1$$

In practise it would probably need to be at least 1.7 : 1.