3F2 Systems and Control: 2011 Solutions

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1. (a) Taking the state vector to be $x = [P_g, P_{c_1}, P_{c_2}, \ldots, P_{c_M}, E]^T$, the state- space equation corresponding to the given equations is:

$$\begin{bmatrix} \dot{P}_g \\ \dot{P}_{c_1} \\ \dot{P}_{c_2} \\ \vdots \\ \dot{P}_{c_M} \\ \dot{E} \end{bmatrix} = \begin{bmatrix} c_g/\tau_g & 0 & 0 & \cdots & 0 & -k/\tau_g \\ 0 & c_c/\tau_c & 0 & \cdots & 0 & 0 \\ 0 & 0 & c_c/\tau_c & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_c/\tau_c & 0 \\ 1 & -1 & -1 & \cdots & -1 & 0 \end{bmatrix} \begin{bmatrix} P_g \\ P_{c_1} \\ P_{c_2} \\ \vdots \\ P_{c_M} \\ E \end{bmatrix} + \begin{bmatrix} 1/\tau_g \\ -1/\tau_c \\ -1/\tau_c \\ \vdots \\ -1/\tau_c \\ 0 \end{bmatrix} p \qquad (1)$$

(b) Equilibrium is defined by $\dot{P}_g = \dot{P}_{c_i} = \dot{E} = 0$. From equation (2) in the question this requires $\boxed{P_{c_i} = p/c_c}$ for each i. From equation (3) it then follows that, at equilibrium, $\boxed{P_g = Mp/c_c}$. Finally, from equation (1) we must have $E = (p - c_g P_g)/k$, namely $\boxed{E = p(1 - Mc_g/c_c)/k}$.

(c) In the case $M = 1$ the state-space equation becomes

$$\begin{bmatrix} \dot{P}_g \\ \dot{P}_{c_1} \\ \dot{E} \end{bmatrix} = \begin{bmatrix} c_g/\tau_g & 0 & -k/\tau_g \\ 0 & c_c/\tau_c & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} P_g \\ P_{c_1} \\ E \end{bmatrix} + \begin{bmatrix} 1/\tau_g \\ -1/\tau_c \\ 0 \end{bmatrix} p \qquad (2)$$

Stability of this system depends on the eigenvalues of the square 3×3 matrix. To simplify notation, write this matrix as

$$A = \begin{bmatrix} a & 0 & -b \\ 0 & c & 0 \\ 1 & -1 & 0 \end{bmatrix} \qquad (3)$$

where $a = c_g/\tau_g$, $b = k/\tau_g$ and $c = c_c/\tau_c$, and we know from the question that $a > 0$, $b > 0$ and $c > 0$. Then the characteristic polynomial of A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - a & 0 & b \\ 0 & \lambda - c & 0 \\ -1 & 1 & \lambda \end{vmatrix} \qquad (4)$$

$$= (\lambda - a)[\lambda(\lambda - c)] + b(0 + \lambda - c) \qquad (5)$$

$$= (\lambda - c)(\lambda^2 - a\lambda + b) \qquad (6)$$

Thus one eigenvalue is at $c > 0$, and hence the system is unstable. (In fact the other two eigenvalues are also in the right half-plane, as can be deduced from the pattern of signs of the coefficients in the quadratic factor.)
Alternative: You could multiply out the cubic polyomial and apply the Routh- Hurwitz test. But that is unnecessary work.

(d) We now have

$$p = \begin{bmatrix} -\alpha & 0 & -\beta \end{bmatrix} \begin{bmatrix} P_g \\ P_{c_1} \\ E \end{bmatrix} \tag{7}$$

hence we have

$$\begin{bmatrix} \dot{P}_g \\ \dot{P}_{c_1} \\ \dot{E} \end{bmatrix} = \left\{ \begin{bmatrix} c_g/\tau_g & 0 & -k/\tau_g \\ 0 & c_c/\tau_c & 0 \\ 1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 1/\tau_g \\ -1/\tau_c \\ 0 \end{bmatrix} \begin{bmatrix} -\alpha & 0 & -\beta \end{bmatrix} \right\} \begin{bmatrix} P_g \\ P_{c_1} \\ E \end{bmatrix} \tag{8}$$

$$= \left\{ \begin{bmatrix} c_g/\tau_g & 0 & -k/\tau_g \\ 0 & c_c/\tau_c & 0 \\ 1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} -\alpha/\tau_g & 0 & -\beta/\tau_g \\ \alpha/\tau_c & 0 & \beta/\tau_c \\ 0 & 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} P_g \\ P_{c_1} \\ E \end{bmatrix} \tag{9}$$

$$= \begin{bmatrix} (c_g-\alpha)/\tau_g & 0 & -(k+\beta)/\tau_g \\ \alpha/\tau_c & c_c/\tau_c & \beta/\tau_c \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} P_g \\ P_{c_1} \\ E \end{bmatrix} \tag{10}$$

If $\beta = -k$ then the square matrix is 'block-triangular':

$$\begin{bmatrix} \dot{P}_g \\ \dot{P}_{c_1} \\ \dot{E} \end{bmatrix} = \begin{bmatrix} (c_g-\alpha)/\tau_g & 0 & 0 \\ \alpha/\tau_c & c_c/\tau_c & \beta/\tau_c \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} P_g \\ P_{c_1} \\ E \end{bmatrix} \tag{11}$$

and hence one of its eigenvalues is $(c_g - \alpha)/\tau_g$, while the other two are the eigenvalues of the bottom-right 2×2 block:

$$\begin{bmatrix} c_c/\tau_c & \beta/\tau_c \\ -1 & 0 \end{bmatrix}$$

The determinant of this block is $\beta/\tau_c < 0$, since $\beta = -k < 0$. But the determinant is the product of the eigenvalues; hence the two eigenvalues must be of opposite signs, so one of them must be in the right half-plane. Hence the system is unstable.
Alternative: Find the characteristic polynomial of the block: $\lambda(\lambda - c_c/\tau_c) + \beta/\tau_c$. *The coefficients are not all of the same sign, so at least one of its roots must be in the right half-plane.*

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2. (a) The transfer function of the car is $G(s) = 1/(ms+b)$. The appropriate block diagram is shown in fig.1.

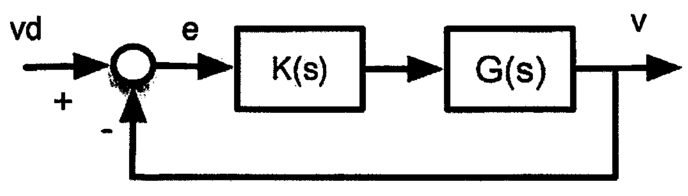

Figure 1: Block diagram of cruise control system in Q.2(a).

Write the PI controller as $K(s) = k(s+1)/s$, so the return-ratio is

$$L(s) = \frac{k(s+1)}{s(ms+b)} = \frac{k}{m}\frac{s+1}{s(s+\frac{b}{m})} = c\frac{s+1}{s(s+\frac{b}{m})} \tag{12}$$

where $c = k/m$. (The point here is to put the transfer into 'monic' form, which will help to get part (d) correct.)
So there are poles at 0 and at $-b/m = -1/20$, and a zero at -1. The root-locus thus has 2 branches, which start at the two poles; one of them terminates at -1, while the other one goes to ∞ — it

must do so along the negative real axis (the angle between the asymptote and the positive real axis is $\pi/(2-1)$). The root-locus is on the real axis for $-1/20 \le s \le 0$ and for $s \le -1$ (to the left of an odd number of poles and zeros). There must therefore be a breakaway point between the two poles, and another one to the left of the zero.

A sketch of the root-locus diagram must therefore look something like fig.2. (The portion off the real axis is in fact a circle, but that is not expected here. Also one could look ahead to part (b)(ii) of the question to locate the breakaway points accurately, but again that is not important here.)

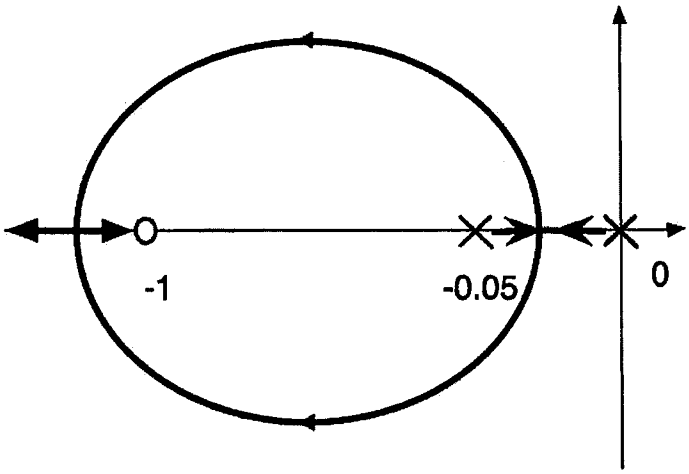

Figure 2: Sketch of root-locus diagram for Q.2(a) — not to scale!

(b) i. Write the closed-loop characteristic equation $1+L(s)=0$ as a polynomial equation $d(s)+n(s)=0$, where $L(s)=n(s)/d(s)$. At a breakaway point s_0 this equation has a repeated root, so it must be possible to write

$$d(s)+n(s)=(s-s_0)^p q(s) \tag{13}$$

where $q(s)$ is some polynomial and $p \ge 2$ (usually $p=2$), or

$$1+L(s)=\frac{(s-s_0)^p q(s)}{d(s)} \tag{14}$$

Differentiating this with respect to s gives

$$\frac{dL(s)}{ds}=\frac{[p(s-s_0)^{p-1}q(s)+(s-s_0)^p q'(s)]d(s)-(s-s_0)^p q(s)d'(s)}{d(s)^2} \tag{15}$$

and therefore

$$\left.\frac{dL(s)}{ds}\right|_{s=s_0}=0 \tag{16}$$

since $(s-s_0)$ is a factor of $dL(s)/ds$.

ii. From (a) we get

$$\frac{dL(s)}{ds} = c\frac{s(s+\frac{b}{m})-(s+1)(2s+\frac{b}{m})}{[s(s+\frac{b}{m})]^2} \tag{17}$$

$$= c\frac{-s^2-2s-\frac{b}{m}}{[\cdots]^2} \tag{18}$$

$$= -c\frac{s^2+2s+\frac{1}{20}}{[\cdots]^2} \tag{19}$$

so the breakaway points are roots of $s^2+2s+0.05=0$. But

$$(s+1.9747)(s+0.0253) \equiv s^2+2s+0.05 \tag{20}$$

so the values given are verified to be breakaway points.

(c) Placing both poles at -0.0253 would give both time constants values of $1/0.0253 \approx 40$ sec, which is clearly much slower than the specification. Placing both poles at -1.9747 would give time constants of $1/1.9747 \approx 0.5$ sec, which satisfies specification (ii). Since both poles are real and coincident, the damping factor would be $\zeta = 1$, which satisfies specification (i). Since a PI controller is used, the steady-state error will be 0, which satisfies specification (iii).

(d) From part (a), the closed-loop poles are the solutions of

$$1 + L(s) = 1 + c\frac{s+1}{s(s+\frac{b}{m})} = 0 \tag{21}$$

But we want $s = -1.9747$ to be a solution (and we know from part (b) that it is for some positive value of c). Therefore the required value of c is given by

$$c = \frac{|-1.9747| \times |-1.9747 + 0.05|}{|-1.9747+1|} = \frac{1.9747 \times 1.9247}{0.9747} = 3.9 \tag{22}$$

But $c = k/m$, hence the required value is $\boxed{k = 3900}$.

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3. (a) Given $\dot{x} = Ax + Bu$ and $y = Cx + Du$, Laplace transform the first equation to get $sIX(s) - x(0) = AX(s) + BU(s)$ where $X(s)$ is the Laplace transform of $x(t)$ etc. This gives $(sI - A)X(s) = BU(s) + x(0)$. Assume that $x(0) = 0$ (standard assumption when finding transfer functions), then $X(s) = (sI - A)^{-1}BU(s)$. Now taking the Laplace transform of y gives

$$Y(s) = CX(s) + DU(s) = C(sI - A)^{-1}BU(s) + DU(s) \tag{23}$$

and hence the transfer function is

$$G(s) = C(sI - A)^{-1}B + D \tag{24}$$

(b) i. We have

$$u_2 = w = C_2x_2 \quad \text{and} \tag{25}$$
$$v = y = C_1x_1 + D_{11}u_1 + D_{12}C_2x_2 \tag{26}$$

Hence

$$\dot{x}_1 = A_1x_1 + B_{12}C_2x_2 + B_{11}u_1 \tag{27}$$
$$\dot{x}_2 = B_2C_1x_1 + (A_2 + B_2D_{12}C_2)x_2 + B_2D_{11}u_1 \tag{28}$$

or, in standard form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & B_{12}C_2 \\ B_2C_1 & A_2 + B_2D_{12}C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_2D_{11} \end{bmatrix} u_1 \tag{29}$$

This gives A and B as required. From (26) we have

$$C = \begin{bmatrix} C_1 & D_{12}C_2 \end{bmatrix} \quad \text{and} \quad D = D_{11} \tag{30}$$

ii. A system with realisation (A, B, C, D) is controllable if and only if the controllability matrix

$$P = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \tag{31}$$

has rank n, where n is the state dimension.
With the given data we have

$$A = \begin{bmatrix} 1 & 1 \\ \beta & \alpha \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{32}$$

with $n = 2$, and hence

$$P = \begin{bmatrix} 1 & 1 \\ 0 & \beta \end{bmatrix} \tag{33}$$

Thus the system S is controllable if and only if $\beta \neq 0$.
For the system S_2 we have $n_2 = 1$ and hence $P = B_2 = \beta$, so S_2 is controllable if and only if $\beta \neq 0$.
Thus S is controllable if and only if S_2 is controllable.

iii. With the given data, we have $C = [1 \quad 0]$ and $D = 0$, so:

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-1 & -1 \\ -\beta & s-\alpha \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \tag{34}$$

$$= \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-\alpha & 1 \\ \beta & s-1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{(s-1)(s-\alpha)-\beta} \tag{35}$$

$$= \begin{cases} \frac{s-\alpha}{s^2-(1+\alpha)s+\alpha-\beta} & \text{if } \beta \neq 0, \\ \frac{1}{s-1} & \text{if } \beta = 0. \end{cases} \tag{36}$$

Thus we have a second-order transfer function when S is controllable — as expected since $n = 2$ — but only first-order when S is not controllable. Note also that in the latter case the pole of the transfer function is the same as the pole of S_1, whereas in the former case both poles are shifted by the feedback through S_2.

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4. (a) Bookwork. Standard block diagram.
A state observer is a system which has u and y as inputs, and $\hat{x}$ as output, such that $\hat{x}$ is an estimate of the state x of a system described by the state-space equations $\dot{x} = Ax + Bu$, $y = Cx + Du$. $\hat{x}$ is an estimate of x in the sense that $x - \hat{x} \to 0$ as $t \to \infty$. A state observer consists of a copy of the system being observed, with additional feedback of the error $x - \hat{x}$ through a gain matrix L.

(b) From the block diagram, the equations governing the system being observed and the state observer are

$$\frac{dx}{dt} = Ax + Bu \tag{37}$$

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - \hat{y}) \tag{38}$$

where $\hat{y} = C\hat{x} + Du$. Defining the state estimation error $e = x - \hat{x}$ and subtracting these equations gives

$$\frac{de}{dt} = (A - LC)e \tag{39}$$

The behaviour of the estimation error e is thus governed by the matrix $A - LC$. In particular, the eigenvalues of this matrix will determine whether the observer is stable, ie whether e converges to 0, and how fast it does so.
Now the eigenvalues of a matrix are not changed if it is transposed, so the eigenvalues of $A - LC$ are the same as those of $(A - LC)^T = A^T - C^T L^T$. So if the desired eigenvalue locations for $A - LC$ are stored in `poles` then the algorithm `place` can be called as

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place(A',C',poles)
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to give L^T, and the result transposed to get L.
Condition: For this to succeed, the system being observed must be observable.

(c) To estimate T_L using a state observer, consider T_L to be an additional state variable, with governing equation $\dot{T}_L = 0$, that is assume that T_L is constant. Combining this with the equation given for ω leads to the state-space description

$$\begin{bmatrix} \dot{\omega} \\ \dot{T}_L \end{bmatrix} = \begin{bmatrix} -B/J & -1/J \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \omega \\ T_L \end{bmatrix} + \begin{bmatrix} 1/J \\ 0 \end{bmatrix} T \tag{40}$$

$$\omega = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \omega \\ T_L \end{bmatrix} \tag{41}$$

The observability matrix for this system is

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -B/J & -1/J \end{bmatrix} \tag{42}$$

which is non-singular, so the system is observable.
It is therefore possible to design a stable observer for this system which will give $\hat{T}_L$ as one of its outputs, such that $T_L - \hat{T}_L \to 0$ as $t \to \infty$.

(d) Choosing L such that fast convergence of $\hat{T}_L$ to T_L is obtained is likely to require large numerical values in L. If the measurements of ω are noisy then the noise will be amplified by these large values, and will result in erratic estimates of T_L. In this case it will be desirable to allow slower convergence of the estimate, which will allow the estimate to be smoothed, as more weight is put on the dynamics built in to the model, and less weight on individual measurements. A compromise will be needed, so that the speed of convergence remains high enough for the estimate to be useful. The more noise on the measurements, the slower the appropriate convergence speed. Also, as the noise level increases, the measurement of ω becomes less reliable, so that the estimate $\hat{\omega}$ obtained from the observer becomes of interest; the design will need to ensure that this also converges quickly enough to be useful, without being too noisy.
