ENGINEERING TRIPOS PART IIA

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Thursday 28 April 2011 9.00 to 12.00

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Module 3A1

FLUID MECHANICS I

*Answer not more than five questions.*

*All questions carry the same number of marks.*

*The* ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

*Attachments:*

- *3A1 Data Sheet for Applications to External Flows (2 pages);*
- *Boundary Layer Theory Data Card (1 page);*
- *Potential Flow Data Sheet (2 pages).*

STATIONERY REQUIREMENTS
Single-sided script paper.

SPECIAL REQUIREMENTS
Engineering Data Book.
CUED approved calculator allowed.

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator**

Version: 1

1 A two-dimensional inviscid, incompressible flow is produced by a line source of strength $m$, located at point $(0, a)$, and a line sink of strength $m$, located at point $(0, -a)$, as shown in Fig. 1a.

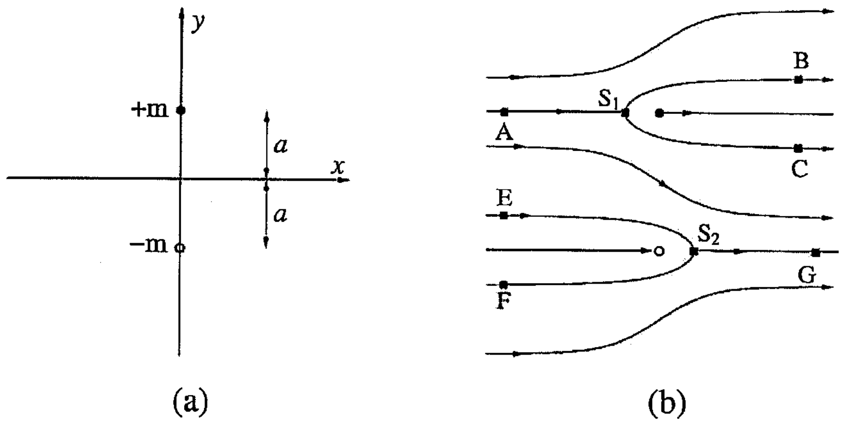



Fig. 1

(a) A uniform flow of speed $U$ from left to right is added. Write down the complex potential for this flow and find the positions of the two stagnation points. [20%]

(b) For the case where $m/(\pi a U)$ is very much smaller than 1 but still positive, explain why the streamline pattern should look like the sketch in Fig. 1b. Find the difference of the streamfunction values $\psi_B - \psi_C$ and $\psi_E - \psi_F$, where points $B$, $C$, $E$ and $F$ are as indicated in Fig. 1b. [20%]

(c) As $m/(\pi a U)$ is gradually increased, the streamlines $AS_1C$ and $ES_2G$ approach each other, where $S_1$ and $S_2$ are the two stagnation points indicated in Fig. 1b. Sketch the streamline pattern when they just merge, and find $\psi_A - \psi_F$ for this case. [20%]

(d) Sketch the streamline pattern when $m/(\pi a U)$ is very much larger than 1. [20%]

(e) Determine whether any of the flow emanating from the source is captured by the sink for the case when $a = U = 1$ and $m = \pi$. [20%]

2 (a) Explain what is meant by *circulation* and the role that it plays in explaining lift on a body in a two-dimensional incompressible, inviscid flow. [15%]

(b) For polar coordinates $r$ and $\theta$, show that the stream function $\psi$ given by:

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin(\theta - \alpha) - \frac{\Gamma}{2\pi}\ln r$$

where $\Gamma$, $U$ and $\alpha$ are positive constants, can represent flow around a circular cylinder and identify the direction of the flow at large distances from the cylinder. [15%]

(c) For the case $0 < \Gamma < 4\pi a U$, show that there are two stagnation points on the cylinder and sketch the streamlines of the flow. [30%]

(d) Find the lift force on the cylinder and identify clearly on a diagram the direction in which it acts. [20%]

(e) Discuss in what ways high Reynolds number viscous flow would deviate from the inviscid solution obtained above. [20%]

3 (a) Explain what is meant by *vortex stretching*. Explain how Kelvin's Circulation Theorem and Stokes' Theorem applied to vorticity can be used to calculate the vorticity changes due to vortex stretching in an incompressible, inviscid fluid. [25%]

(b) A diffuser with area ratio 2 has a constant height $h$ in the $y$-direction, but varies with width in the $z$-direction as shown in Fig. 2. The entry flow to the diffuser has a velocity profile:

$$u = U\left(1 + \frac{y}{3h}\right)$$

where $U$ is a constant. The flow through the diffuser may be treated as incompressible and inviscid. The $y$ and $z$ components of the velocity are very small and can be neglected. Determine the velocity profile at the exit and the pressure rise across the diffuser. [40%]

(c) If the design of the diffuser is changed to increase the outlet width in the $z$-direction while keeping the inlet width and the height unchanged, for what value of area ratio would you expect the flow to separate even when the area variation is very gradual. [35%]

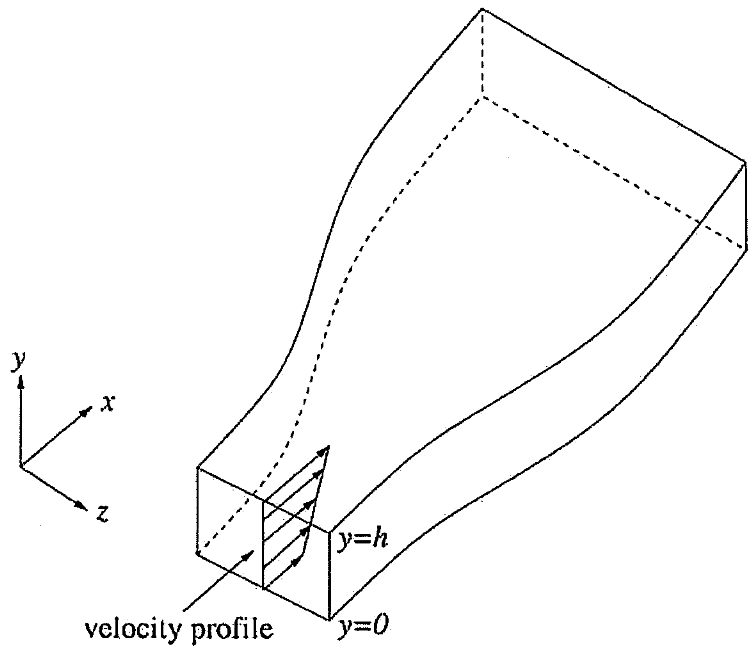



Fig. 2

4 An incompressible flow at high Reynolds number past a flat plate is sketched in Fig. 3. The sharp edge of the plate is at $(x,y) = (0,0)$. The fluid exerts a drag force on the plate due to the no-slip condition. The velocity distribution $u(y)$ at any particular downstream position $x$ shows a smooth drop-off to zero at the wall. The velocity just upstream of the sharp edge is uniform and parallel, $u = U =$ constant, and the flow is steady.

(a) By considering the conservation of mass, derive an expression for the boundary-layer displacement thickness $\delta^*(x)$ at $x$. [20%]

(b) The pressure gradient is zero in this flow. By considering the conservation of momentum, derive an expression for the momentum thickness $\theta(x)$ at $x$ and relate it to the drag force per unit width from 0 to $x$ on one side of the plate. [40%]

(c) If $\tau_w(x)$ is the local shear stress on the plate, write down the total drag per unit width $D$ on one side of the plate of length $L$. [10%]

(d) The friction coefficient and drag coefficient are defined respectively as:

$$C_f(x) = \frac{\tau_w(x)}{\frac{1}{2}\rho U^2} \quad \text{and} \quad C_D = \frac{D}{\frac{1}{2}\rho U^2 L}$$

where $\rho$ is the density of the fluid. Derive expressions for these two coefficients in terms of the momentum thickness $\theta(x)$. [30%]

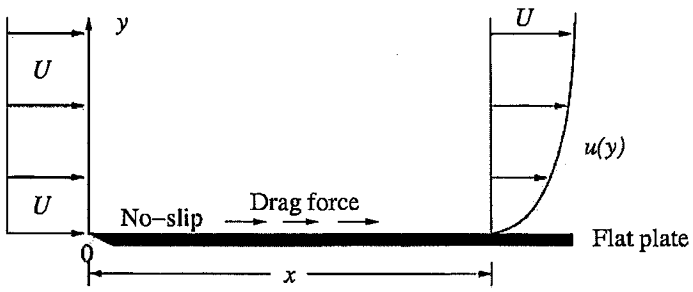



Fig. 3

5 A steady laminar boundary layer on a flat plate has a fourth-order polynomial velocity profile $u$:

$$\frac{u}{U} = a_0 + a_1\left(\frac{y}{\delta}\right) + a_2\left(\frac{y}{\delta}\right)^2 + a_3\left(\frac{y}{\delta}\right)^3 + a_4\left(\frac{y}{\delta}\right)^4$$

where $U$ is the free stream velocity, $a_0$, ..., $a_4$ are constants and $\delta$ is a measure of the boundary layer thickness.

(a) Three boundary conditions for the velocity profile are: $u(0) = 0$, $u(\delta) = U$ and $\partial u/\partial y = 0$ at $y = \delta$. Explain the underlying physical meaning of these conditions. [10%]

(b) The free stream velocity $U$ is constant, i.e., $dU/dx = 0$. Derive the two additional boundary conditions: $\partial^2 u/\partial y^2 = 0$ at $y = 0$ and $y = \delta$. [20%]

(c) Use the above five boundary conditions to determine the five constants $a_0$, ..., $a_4$, and hence the velocity profile. [20%]

(d) Find the momentum thickness $\theta$ and the local skin friction coefficient $C_f'$. [20%]

(e) Write down the momentum integral equation in terms of the quantities you found in part (d). [10%]

(f) Find the variation of $\delta$ in terms of $Re_x = Ux/\nu$, where $x$ is the distance from the leading edge of the plate (where $\delta = 0$) and $\nu$ is the kinematic viscosity. [20%]

6 (a) Describe the three component flows which are superposed in the classical two-dimensional thin-airfoil theory. [10%]

(b) Derive a relationship between the chordwise distribution of vorticity and the camber line slope explaining carefully any approximations made. [20%]

(c) Derive a relationship for the airfoil lift coefficient in terms of the vorticity distribution. [10%]

(d) Hence show that:

$$C_L = 2\pi\alpha + \pi\left(g_0 + \frac{1}{2}g_1\right)$$

where the symbols are defined in the Data Sheet and describe the physical meaning of each term. You should note that:

$$\int_0^\pi \sin m\theta \sin n\theta \, d\theta = 0 \quad \text{for } m \neq n$$

[20%]

(e) Use the result of part (d) to show how the difference of camber is weighted towards the rear of the airfoil. [20%]

(f) Explain why the leading term (related to $g_0$), which is different from the other Fourier series terms, is needed for the Camber solution as shown in the Data Sheet. [20%]

7 A researcher wants to study the wake behind a bluff body in an open-loop wind-tunnel. The rear of the body can be simplified as a circular cylinder of 20 mm diameter, aligned with the flow direction. The free stream velocity is 20 m s$^{-1}$, and the density of air is 1.225 kg m$^{-3}$.

(a) Sketch a basic open-loop wind-tunnel and identify its key components. [20%]

(b) The wind-tunnel has a closed working section of 0.5 m $\times$ 0.5 m cross-section and a total length of 5 m, of which 3 m is allocated to the diffuser. Estimate the minimum tunnel power requirement and determine the power factor. Identify any key losses not included in your estimate. [20%]

(c) The wake flow is to be investigated using two-dimensional PIV. The region of interest is indicated in Fig. 4. Make a sketch of the PIV set-up identifying the key components and show where they are placed relative to the bluff body and the working section. [20%]

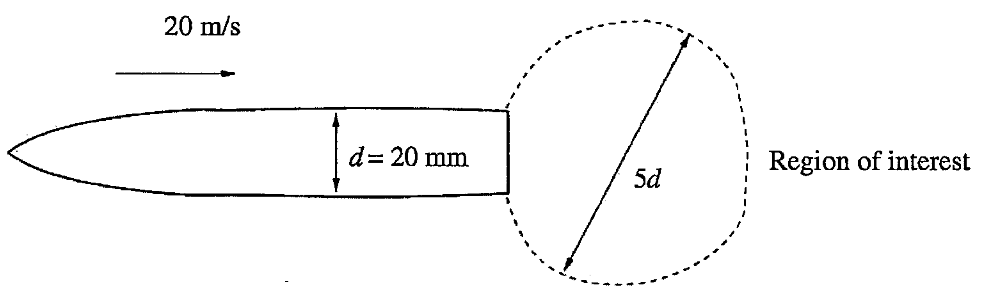



Fig. 4

(d) The camera has a resolution of 1024 $\times$ 1024 pixels. Using sensible parameters for the PIV set-up, estimate :

(i) the size of the light sheet;

(ii) the spatial resolution of the measurements;

(iii) the interframe time;

(iv) the maximum and minimum velocities resolved.

[20%]

(e) Given that the cylinder has a streamwise length of 200 mm, do you think you can use the above measurement system to determine the shape of the boundary layer profile growing on the bluff body upstream of the trailing edge? Justify your answer. [20%]

8 (a) Describe the use of horseshoe vortices in Lifting-Line Theory. [20%]

(b) Derive expressions for the downwash and the downwash angle in Lifting-Line Theory in terms of the spanwise circulation distribution. [20%]

(c) Explain the physical origin of the induced drag. Explain the key steps in the derivation of an expression for it in terms of the spanwise circulation distribution and the downwash angle. [20%]

(d) Starting from the expressions for the Wing lift and the Induced drag given in the Data Sheet, derive the following:

$$C_{Di} = (1+\delta)\frac{C_L^2}{\pi A_R}$$

and hence show that an elliptic lift distribution has the minimum induced drag. You may use the following results:

$$\int_0^{\pi} \sin m\theta \sin n\theta \, d\theta = \frac{\pi}{2} \quad \text{if } n = m$$

$$\int_0^{\pi} \sin m\theta \sin n\theta \, d\theta = 0 \quad \text{if } n \neq m$$

[40%]

**END OF PAPER**

# 3A1 Data Sheet for Applications to External Flows

## Aerodynamic Coefficients

For a flow with free-stream density, ρ, velocity U and pressure $p_\infty$:

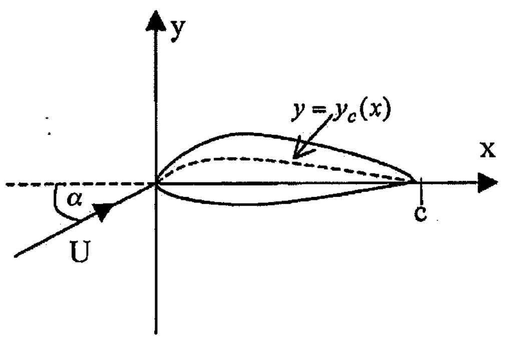
Pressure coefficient: $$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2}$$

Section lift and drag coefficients: $$c_l = \frac{\text{lift}\,(N/m)}{\frac{1}{2}\rho U^2 c},\quad c_d = \frac{\text{drag}\,(N/m)}{\frac{1}{2}\rho U^2 c}$$ (section chord c)

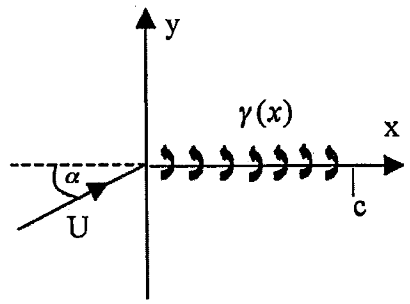
Wing lift and drag coefficients: $$C_L = \frac{\text{lift}\,(N)}{\frac{1}{2}\rho U^2 S},\quad C_D = \frac{\text{drag}\,(N)}{\frac{1}{2}\rho U^2 S}$$ (wing area S)

## Thin Aerofoil Theory

Geometry




Approximate representation




Pressure coefficient: $c_p = \pm\gamma / U$

Pitching moment coefficient: $c_m = (\text{moment about } x = 0) / \frac{1}{2}\rho U^2 c^2$

Coordinate transformation: $x = c(1+\cos\theta)/2 = c\cos^2(\theta/2)$

Incidence solution: $$\gamma = -2U\alpha\frac{1-\cos\theta}{\sin\theta},\quad c_l = 2\pi\alpha,\quad c_m = c_l/4$$

Camber solution: $$\gamma = -U\left[g_0\frac{1-\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} g_n \sin n\theta\right],$$ where

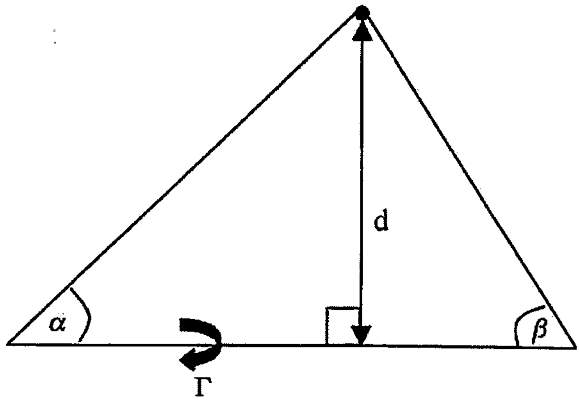
$$g_0 = \frac{1}{\pi}\int_0^\pi\left(-2\frac{dy_c}{dx}\right)d\theta,\quad g_n = \frac{2}{\pi}\int_0^\pi\left(-2\frac{dy_c}{dx}\right)\cos n\theta\, d\theta$$

$$c_l = \pi\left(g_0 + \frac{g_1}{2}\right),\quad c_m = \frac{\pi}{4}\left(g_0 + g_1 + \frac{g_2}{2}\right) = \frac{c_l}{4} + \frac{\pi}{8}(g_1 + g_2)$$

## Glauert Integral

$$\int_0^{\pi} \frac{\cos n\phi}{\cos\phi - \cos\theta}\, d\phi = \pi \frac{\sin n\theta}{\sin\theta}$$

## Line Vortices



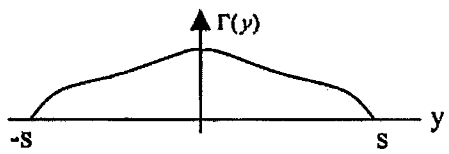

A straight element of circulation $\Gamma$ induces a velocity at P of

$$\frac{\Gamma}{4\pi d}(\cos\alpha + \cos\beta)$$

perpendicular to the plane containing P and the element.

## Lifting-Line Theory

Spanwise circulation distribution:




Aspect ratio: $A_R = 4s^2 / S$

Wing lift: $L = \rho U \int_{-s}^{s} \Gamma(y)\, dy$

Downwash angle: $\alpha_d(y) = \frac{1}{4\pi U} \int_{-s}^{s} \frac{d\Gamma(\eta)/d\eta}{y - \eta}\, d\eta$

Induced drag: $D_i = \rho U \int_{-s}^{s} \Gamma(y) \alpha_d(y)\, dy$

Fourier series for circulation: $\Gamma(y) = Us \sum_{n \text{ odd}} G_n \sin n\theta$, with $y = -s\cos\theta$

Relation between lift and induced drag:

$$C_{Di} = (1+\delta)\frac{C_L^2}{\pi A_R}, \text{ where } \delta = 3\left(\frac{G_3}{G_1}\right)^2 + 5\left(\frac{G_5}{G_1}\right)^2 + \ldots$$

# Module 3A1
# Boundary Layer Theory Data Card

Displacement thickness;

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_1}\right) dy$$

Momentum thickness;

$$\theta = \int_0^\infty \frac{(U_1 - u)u}{U_1^2} dy = \int_0^\infty \left(1 - \frac{u}{U_1}\right) \frac{u}{U_1} dy$$

Energy thickness;

$$\delta_E = \int_0^\infty \frac{(U_1^2 - u^2)u}{U_1^3} dy = \int_0^\infty \left(1 - \left(\frac{u}{U_1}\right)^2\right) \frac{u}{U_1} dy$$

$$H = \frac{\delta^*}{\theta}$$

Prandtl's boundary layer equations (laminar flow);

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{dp_1}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}$$

von Karman momentum integral equation;

$$\frac{d\theta}{dx} + \frac{H+2}{U_1}\theta\frac{dU_1}{dx} = \frac{\tau_o}{\rho U_1^2} = \frac{C_f'}{2}$$

Boundary layer equations for turbulent flow;

$$\begin{aligned} \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} &= \frac{-1}{\rho}\frac{d\overline{p}}{dx} - \frac{\partial \overline{u'v'}}{\partial y} + \nu\frac{\partial^2 \overline{u}}{\partial y^2} \\ \frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} &= 0 \end{aligned}$$

**Module 3A1 Fluid Mechanics I**

# POTENTIAL FLOW DATA SHEET

**Continuity equation** $\nabla \cdot \underline{u} = 0$

**Momentum equation** (inviscid) $\rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{g}$

$\frac{D}{Dt}$ denotes the material derivative, $\frac{\partial}{\partial t} + \underline{u} \cdot \nabla$

**Vorticity** $\underline{\omega} = \nabla \times \underline{u}$

**Vorticity equation** (inviscid) $\frac{D\underline{\omega}}{Dt} = \underline{\omega} . \nabla \underline{u}$

**Kelvin's circulation theorem** (inviscid) $\frac{D\Gamma}{Dt} = 0 \,, \quad \Gamma = \oint \underline{u} . \underline{dl} = \int \underline{\omega} . \underline{dS}$

**For an irrotational flow**

velocity potential ($\phi$) $\underline{u} = \nabla \phi$ and $\nabla^2 \phi = 0$

Bernoulli's equation for inviscid flow,

$\frac{p}{\rho} + \frac{1}{2} V^2 + gz + \frac{\partial \phi}{\partial t}$ = constant throughout flow field, $V = |\underline{u}|$.

## TWO-DIMENSIONAL FLOW

**Streamfunction** ($\psi$)

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad u_\theta = -\frac{\partial \psi}{\partial r}$$

**Lift force** Lift / unit length $= \rho U(-\Gamma)$

**Complex potential** $F(z)$ for irrotational flows, with $z = x + iy$, $F(z) = \phi + i\psi$ and $\frac{dF}{dz} = u - iv$

**Examples of complex potentials**

(i) uniform flow in $x$-direction, $F(z) = Uz$

(ii) source at $z_0$, $F(z) = \frac{m}{2\pi} \ln(z - z_0)$

(iii) doublet at $z_0$, with axis in $x$-direction, $F(z) = \frac{\mu}{2\pi(z - z_0)}$

(iv) anticlockwise vortex at $z_0$, $F(z) = -\frac{i\Gamma}{2\pi} \ln(z - z_0)$

## Module 3A1 Fluid Mechanics I

## TWO-DIMENSIONAL FLOW

**Summary of simple 2-D flow fields**

| | $\phi$ | $\psi$ | circulation | $\underline{u}$ |
|---|---|---|---|---|
| Uniform flow (towards $+x$) | $Ux$ | $Uy$ | 0 | $u = U, v = 0$ |
| Source at origin | $\frac{m}{2\pi}\ln r$ | $\frac{m}{2\pi}\theta$ | 0 | $u_r = \frac{m}{2\pi r}, u_\theta = 0$ |
| Doublet at origin $\theta$ is angle from doublet axis | $\frac{\mu\cos\theta}{2\pi r}$ | $-\frac{\mu\sin\theta}{2\pi r}$ | 0 | $u_r = -\frac{\mu\cos\theta}{2\pi r^2}, u_\theta = -\frac{\mu\sin\theta}{2\pi r^2}$ |
| Anticlockwise vortex at origin | $\frac{\Gamma}{2\pi}\theta$ | $-\frac{\Gamma}{2\pi}\ln r$ | $\Gamma$ around origin | $u_r = 0, \quad u_\theta = \frac{\Gamma}{2\pi r}$ |

## THREE-DIMENSIONAL FLOW

**Summary of simple 3-D flow fields**

| | $\phi$ | **u** |
|---|---|---|
| Source at origin | $-\frac{m}{4\pi r}$ | $u_r = \frac{m}{4\pi r^2}, \quad u_\theta = 0, \quad u_\phi = 0$ |
| Doublet at origin $\theta$ is angle from doublet axis | $\frac{\mu\cos\theta}{4\pi r^2}$ | $u_r = -\frac{\mu\cos\theta}{2\pi r^3}, \quad u_\theta = -\frac{\mu\sin\theta}{4\pi r^3}, \quad u_\phi = 0$ |

October 2010

## Numerics Answers

Jie Li

Quesion 3,
(c) Area ratio is $\sqrt{7}$.
Question 5,
(c) $a_0 = 0$, $a_1 = 2$, $a_2 = 0$, $a_3 = -2$ and $a_4 = 1$.
Question 7,
(b) Power factor $\lambda = 0.3$.