ENGINEERING TRIPOS PART IIA

Tuesday 10 May 2011 9 to 10.30

Module 3C5

DYNAMICS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The **approximate** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
Data sheet: 3C5 Dynamics and 3C6 Vibration (6 pages)

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Final version

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1 A solid cube of side a and mass m is shown in Fig. 1(a). The cube is cut along the plane ABCD as shown to form the triangular prism of mass $m/2$ shown in Fig. 1(b) with its vertices labelled ABCDEF.

(a) Find the principal moments of inertia of the cube. [10%]

(b) For the prism show that it is an "AAC" body and find:

(i) the distance from the plane ABCD to the centre of mass G; [20%]
(ii) the principal moments of inertia through G. [20%]

(c) The prism is free to move in space and is initially spinning with angular velocity ω about an axis perpendicular to the plane ABCD. At a certain instant a small impulsive force I is applied to point A in a direction perpendicular to the plane ABCD so as to cause a small perturbation to the motion of the prism. Find:

(i) the change in velocity of the centre of mass G; [10%]
(ii) the frequency of nutation of the prism; [20%]
(iii) the nutation angle (*ie* the angle of the cone traced out by the spin axis). [20%]

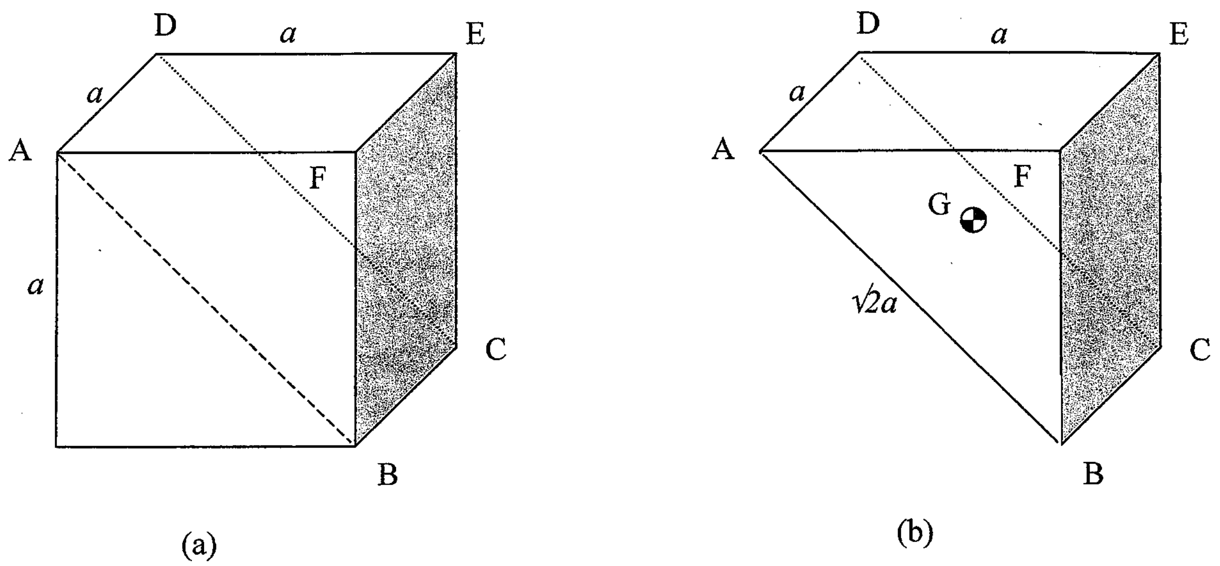

Fig. 1

2 A solid uniform cylinder of radius 300mm, length 750mm and mass 340kg is spinning at 1200rpm about a horizontal light rigid shaft through its axis of symmetry as shown in Fig. 2. The shaft is supported on two spherical bearings (bearings that permit rotation in any direction) separated by a distance of 900mm.

(a) In a certain experiment the spinning cylinder is released from *one* of its bearings and remains supported by the other. Under the action of gravity the cylinder exhibits precession and nutation. After all transients have died away find:

(i) the steady rate of precession; [30%]

(ii) the inclination of the cylinder axis to the horizontal. [30%]

(b) In another experiment an attempt is made to release the spinning cylinder from *both* bearings simultaneously after which the cylinder falls freely under gravity. In practice, however, one bearing releases a short time T after the other. If the time interval T is sufficiently short then a "clean" release occurs, *ie* the cylinder will not wobble appreciably after it is released. Note that the frequency of nutation is $C\omega/A$ for an AAC body spinning at angular velocity ω.

(i) For $T = 0.01$ s show that the cylinder is likely to release cleanly. [20%]

(ii) Find the smallest value of T for which the cylinder will experience the greatest amount of wobble after release. [20%]

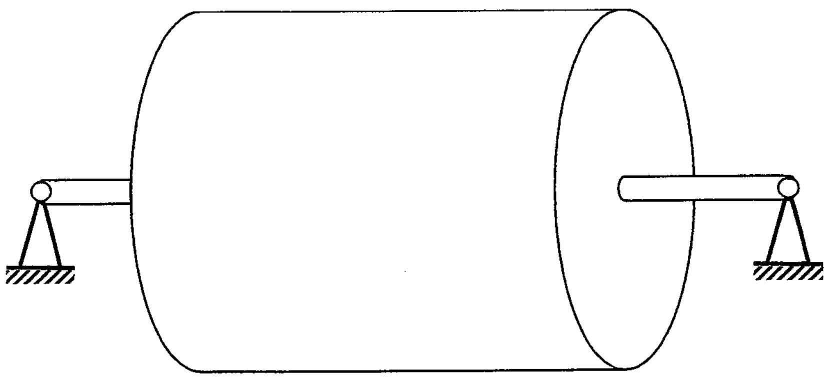

Fig. 2

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3 (a) For a rigid body with principal moments of inertia A, B and C write down an expression for the moment of momentum vector $\boldsymbol{h}$ when the body is spinning with angular velocity $\boldsymbol{\omega} = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k}$ where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is a body-fixed reference frame aligned with the principal axes. Hence derive Euler's equations of motion for the angular motion for the body when subject to a couple $\boldsymbol{Q} = Q_1 \mathbf{i} + Q_2 \mathbf{j} + Q_3 \mathbf{k}$. [40%]

(b) The body is spinning freely in space with steady angular velocity $\boldsymbol{\omega} = \Omega \mathbf{i}$. Use Euler's equations to show that the motion is unstable for the cases $B < A < C$ and $C < A < B$. [50%]

(c) The solid rigid body shown in Fig. 3 is shown roughly to scale. Determine the stability of spin about each of the axes A, B and C . [10%]

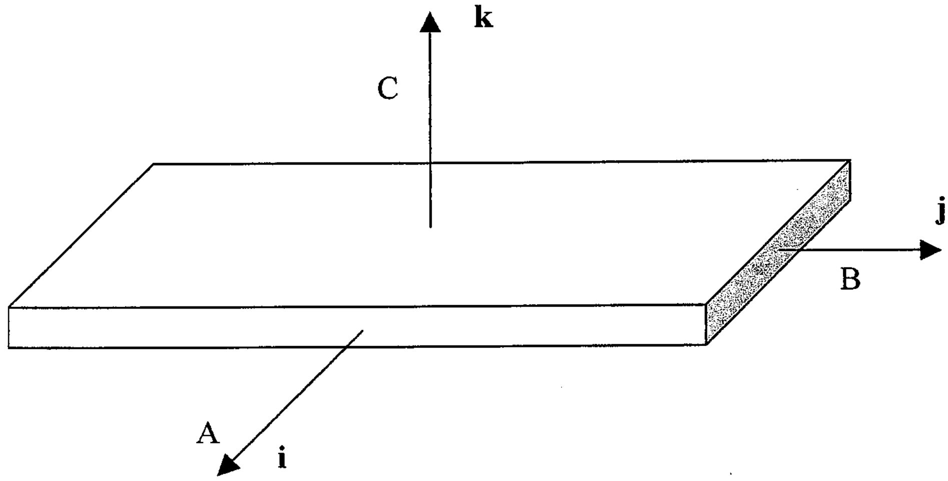

Fig. 3

4 A simplified model of a centrifugal governor is shown in Fig. 4. The system comprises four light bars each of length a arranged to form a planar mechanism, so that points A and B move vertically on a shaft (not shown) separated by a spring of stiffness k. Points C and D are constrained to move in a horizontal plane and each carries a particle of mass M. The motion of the mechanism is described by the internal angle θ and the angle of rotation ϕ about the spin axis AB. The unstretched length of the spring is $2a\sin\theta_0$, so that the spring carries no force when θ has the value θ_0. The mechanism is subject to a torque Q about the spin axis AB.

(a) By using Lagrange's equation, show that the equations of motion of the governor using the rotations θ and ϕ as generalised coordinates are

$$M\ddot{\theta}\sin^2\theta + M\dot{\theta}^2\sin\theta\cos\theta + M\dot{\phi}^2\sin\theta\cos\theta + 2k\cos\theta(\sin\theta - \sin\theta_0) = 0$$

and

$$\frac{d}{dt}\left[2Ma^2\cos^2\theta\dot{\phi}\right] = Q\ .$$

Give a physical interpretation of the generalised momentum in the ϕ coordinate. [30%]

(b) The torque Q is controlled to give a constant rotation rate Ω around the axis AB, so that $\dot{\phi} = \Omega$. Show that in steady equilibrium $\theta = \theta_e$ satisfies the equation:

$$\sin\theta_e = \frac{2k\sin\theta_0}{M\Omega^2 + 2k}\ .$$ [30%]

(c) For $\Omega = 0$ find the natural frequency of vibration about the equilibrium position and describe briefly how you would find the natural frequency when $\Omega \neq 0$. [40%]

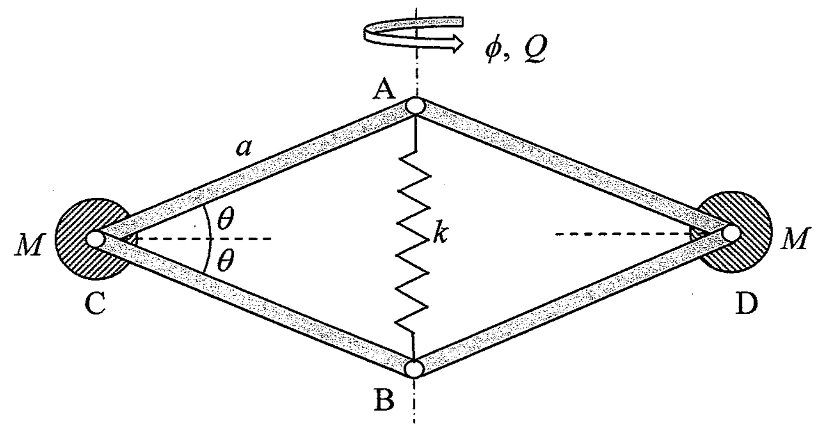

Fig. 4

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5 An N degree of freedom dynamic system is described by generalised coordinates q_j, $j = 1,2,..,N$. It is found that the kinetic and potential energies have the form:

$$T = \sum_{n=1}^{N} \sum_{m=1}^{N} \{A_{nm}\dot{q}_n\dot{q}_m + B_{nm}\dot{q}_n q_m + C_{nm}q_n q_m\}$$

and

$$V = \sum_{n=1}^{N} \sum_{m=1}^{N} D_{nm}q_n q_m$$

where the coefficients A_{nm}, C_{nm}, and D_{nm} are all symmetric (*ie* $A_{nm} = A_{mn}$ etc.), but this is not necessarily true of B_{nm}.

(a) By using Lagrange's equation show that the equations of motion of the system can be written in the form

$$\sum_{n=1}^{N} M_{jn}\ddot{q}_n + \sum_{n=1}^{N} L_{jn}\dot{q}_n + \sum_{n=1}^{N} K_{jn}q_n = Q_j, \qquad j = 1,2,...,N$$

or in the equivalent matrix form

$$\mathbf{M\ddot{q}} + \mathbf{L\dot{q}} + \mathbf{Kq} = \mathbf{Q}$$

where $\mathbf{Q}$ is the vector of generalised forces, and the entries of the matrices **M**, **L**, and **K** can be expressed in terms of the coefficients A_{nm}, B_{nm}, C_{nm}, and D_{nm}. [50%]

(b) Show that the matrices **M** and **K** are symmetric, and that the matrix **L** is skew symmetric (*ie* $L_{nm} = -L_{mn}$). [20%]

(c) The matrix **L** arises in rotating systems, where it is known as the *gyroscopic damping matrix*. The term "damping" is potentially misleading since the matrix **L** cannot be associated with any loss of energy. To demonstrate this, show that the power lost through the gyroscopic damping forces $\mathbf{L\dot{q}}$ is given by

$$\mathbf{\dot{q}}^{\mathrm{T}}\mathbf{L\dot{q}}$$

and hence show that this quantity is zero. Compare this result to the case of conventional damping, for which the associated damping matrix is symmetric. [30%]

END OF PAPER

Part IIA Data sheet

Module 3C5 Dynamics

Module 3C6 Vibration

DYNAMICS IN THREE DIMENSIONS

Axes fixed in direction

(a) Linear momentum for a general collection of particles m_i:

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}^{(e)}$$

where $\boldsymbol{p} = M\,\boldsymbol{v}_G$, M is the total mass, $\boldsymbol{v}_G$ is the velocity of the centre of mass and $\boldsymbol{F}^{(e)}$ the total external force applied to the system.

(b) Moment of momentum about a general point P

$$\boldsymbol{Q}^{(e)} = (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \dot{\boldsymbol{p}} + \dot{\boldsymbol{h}}_G$$
$$= \dot{\boldsymbol{h}}_P + \dot{\boldsymbol{r}}_P \times \boldsymbol{p}$$

where $\boldsymbol{Q}^{(e)}$ is the total moment of external forces about P. Here, $\boldsymbol{h}_P$ and $\boldsymbol{h}_G$ are the moments of momentum about P and G respectively, so that for example

$$\boldsymbol{h}_P = \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_P) \times m_i \dot{\boldsymbol{r}}_i$$
$$= \boldsymbol{h}_G + (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \boldsymbol{p}$$

where the summation is over all the mass particles making up the system.

(c) For a rigid body rotating with angular velocity $\boldsymbol{\omega}$ about a fixed point P at the origin of coordinates

$$\boldsymbol{h}_P = \int \boldsymbol{r} \times (\boldsymbol{\omega} \times \mathbf{r})\, dm = I\,\boldsymbol{\omega}$$

where the integral is taken over the volume of the body, and where

$$I = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \qquad \boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and $A = \int (y^2 + z^2)\,dm$ $\quad B = \int (z^2 + x^2)\,dm$ $\quad C = \int (x^2 + y^2)\,dm$

$D = \int yz\,dm$ $\quad E = \int zx\,dm$ $\quad F = \int xy\,dm$

where all integrals are taken over the volume of the body.

Axes rotating with angular velocity $\boldsymbol{\Omega}$

Time derivatives of vectors must be replaced by the "rotating frame" form, so that for example

$$\dot{\boldsymbol{p}} + \boldsymbol{\Omega} \times \boldsymbol{p} = \boldsymbol{F}^{(e)}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector $\boldsymbol{r}$ is needed, as in 1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

Euler's dynamic equations (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$A\,\dot{\omega}_1 - (B-C)\,\omega_2\,\omega_3 = Q_1$$
$$B\,\dot{\omega}_2 - (C-A)\,\omega_3\,\omega_1 = Q_2$$
$$C\,\dot{\omega}_3 - (A-B)\,\omega_1\,\omega_2 = Q_3$$

where A, B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

(b) Non-body-fixed reference frame for axisymmetric bodies (the "Gyroscope equations"):

$$A\,\dot{\Omega}_1 - (A\,\Omega_3 - C\,\omega_3)\,\Omega_2 = Q_1$$
$$A\,\dot{\Omega}_2 + (A\,\Omega_3 - C\,\omega_3)\,\Omega_1 = Q_2$$
$$C\,\dot{\omega}_3 = Q_3$$

where A, A and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes such that ω_3 and Q_3 are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\boldsymbol{\Omega} = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1=\omega_1$ and $\Omega_2=\omega_2$.

Lagrange's equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy, and Q_i are the non-conservative generalised forces.

VIBRATION MODES AND RESPONSE

Discrete systems

1. The forced vibration of an N-degree-of-freedom system with mass matrix M and stiffness matrix K (both symmetric and positive definite) is

$$M\,\ddot{\underline{y}} + K\,\underline{y} = \underline{f}$$

where $\underline{y}$ is the vector of generalised displacements and $\underline{f}$ is the vector of generalised forces.

2. **Kinetic energy**

$$T = \frac{1}{2}\dot{\underline{y}}^t M \dot{\underline{y}}$$

Potential energy

$$V = \frac{1}{2}\underline{y}^t K \underline{y}$$

3. The natural frequencies ω_n and corresponding mode shape vectors $\underline{u}^{(n)}$ satisfy

$$K\underline{u}^{(n)} = \omega_n{}^2 M\underline{u}^{(n)}\,.$$

4. **Orthogonality and normalisation**

$$\underline{u}^{(j)t} M \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

$$\underline{u}^{(j)t} K \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ \omega_n^2, & j = k \end{cases}$$

5. **General response**

The general response of the system can be written as a sum of modal responses

$$\underline{y}(t) = \sum_{j=1}^{N} q_j(t)\,\underline{u}^{(j)} = U\underline{q}(t)$$

where U is a matrix whose N columns are the normalised eigenvectors $\underline{u}^{(j)}$ and q_j can be thought of as the "quantity" of the jth mode.

Continuous systems

The forced vibration of a continuous system is determined by solving a partial differential equation: see p. 6 for examples.

$$T = \frac{1}{2}\int \dot{u}^2\,dm$$

where the integral is with respect to mass (similar to moments and products of inertia).

See p. 6 for examples.

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see p. 6) and boundary conditions, assuming harmonic time dependence.

$$\int u_j(x)\,u_k(x)\,dm = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

The general response of the system can be written as a sum of modal responses

$$w(x,t) = \sum_j q_j(t)\,u_j(x)$$

where $w(x,t)$ is the displacement and q_j can be thought of as the "quantity" of the jth mode.

6. Modal coordinates $\underline{q}$ satisfy

$$\ddot{\underline{q}} + \left[\operatorname{diag}(\omega_j^2)\right] \underline{q} = \underline{Q}$$

where $\underline{y} = U\underline{q}$ and the modal force vector

$$\underline{Q} = U^t \underline{f} \ .$$

Each modal amplitude $q_j(t)$ satisfies

$$\ddot{q}_j + \omega_j^2 q_j = Q_j$$

where $Q_j = \int f(x,t)\, u_j(x)\, dm$ and $f(x,t)$ is the external applied force distribution.

7. Frequency response function

For input generalised force f_j at frequency ω and measured generalised displacement y_k the transfer function is

$$H(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For force F at frequency ω applied at point x, and displacement w measured at point y, the transfer function is

$$H(x,y,\omega) = \frac{w}{F} = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x,y,\omega) = \frac{w}{F} \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2 + 2i\,\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

8. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)} u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with low modal overlap, if the factor $u_n(x)\, u_n(y)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

9. Impulse response

For a unit impulsive generalised force $f_j = \delta(t)$ the measured response y_k is given by

$$g(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin\omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin\omega_n t\, e^{-\omega_n\zeta_n t}$$

for $t \geq 0$ (with small damping).

For a unit impulse applied at $t = 0$ at point x, the response at point y is

$$g(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin\omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(x,y,t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin\omega_n t\, e^{-\omega_n\zeta_n t}$$

for $t \geq 0$ (with small damping).

10. Step response

For a unit step generalised force $f_j = \begin{cases} 0 & t<0 \\ 1 & t \ge 0 \end{cases}$ the measured response y_k is given by

$$h(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos\omega_n t\right]$$

for $t \ge 0$ (with no damping), or

$$h(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos\omega_n t \, e^{-\omega_n \zeta_n t}\right]$$

for $t \ge 0$ (with small damping).

For a unit step force applied at $t=0$ at point x, the response at point y is

$$h(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2} \left[1 - \cos\omega_n t\right]$$

for $t \ge 0$ (with no damping), or

$$h(t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2} \left[1 - \cos\omega_n t \, e^{-\omega_n \zeta_n t}\right]$$

for $t \ge 0$ (with small damping).

Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is $\frac{V}{\tilde{T}} = \frac{\underline{y}^t K \underline{y}}{\underline{y}^t M \underline{y}}$ where $\underline{y}$ is the vector of generalised coordinates, M is the mass matrix and K is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions on p. 6.

If this quantity is evaluated with any vector $\underline{y}$, the result will be

(1) $\ge$ the smallest squared frequency;

(2) $\le$ the largest squared frequency;

(3) a good approximation to ω_k^2 if $\underline{y}$ is an approximation to $\underline{u}^{(k)}$.

(Formally, $\frac{V}{\tilde{T}}$ is *stationary* near each mode.)

Governing equations for continuous systems

Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $w(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\frac{\partial^2 w}{\partial t^2} - P\frac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \frac{1}{2}P\int\left(\frac{\partial w}{\partial x}\right)^2 dx$	$T = \frac{1}{2}m\int\left(\frac{\partial w}{\partial t}\right)^2 dx$

Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $f(x,t)$ per unit length.
Polar moment of area is $J = (\pi/2)\left(a^4 - b^4\right)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\frac{\partial^2 \theta}{\partial t^2} - GJ\frac{\partial^2 \theta}{\partial x^2} = f(x,t)$	$V = \frac{1}{2}GJ\int\left(\frac{\partial \theta}{\partial x}\right)^2 dx$	$T = \frac{1}{2}\rho J\int\left(\frac{\partial \theta}{\partial t}\right)^2 dx$

Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $w(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\frac{\partial^2 w}{\partial t^2} - EA\frac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \frac{1}{2}EA\int\left(\frac{\partial w}{\partial x}\right)^2 dx$	$T = \frac{1}{2}\rho A\int\left(\frac{\partial w}{\partial t}\right)^2 dx$

Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $w(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\frac{\partial^2 w}{\partial t^2} + EI\frac{\partial^4 w}{\partial x^4} = f(x,t)$	$V = \frac{1}{2}EI\int\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx$	$T = \frac{1}{2}\rho A\int\left(\frac{\partial w}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.