ENGINEERING TRIPOS PART IIA

Thursday 12 May 2011 9 to 10.30

Module 3C6

VIBRATIONS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
Data Sheet: 3C5 Dynamics and 3C6 Vibration (6 pages)

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Final Version

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1. The Ariane 5 launch vehicle can be modelled approximately as an undamped vertical rod of length L, cross-sectional area A, density ρ and Young's modulus E. The ends of the rod are free from constraint.

(a) Write down the governing differential equation and boundary conditions for axial vibration of the rod, and show that the natural frequencies are given by

$$\omega_n = \left(\frac{n\pi}{L}\right)\sqrt{\frac{E}{\rho}}, \qquad n = 0,1,2,3,\ldots$$ [30%]

(b) Derive expressions for the mode shapes of the rod, and normalise each mode shape to unit generalised mass. Note carefully the normalisation factor required for the rigid body mode corresponding to $n = 0$. [20%]

(c) The firing of the engines at launch can be modelled as a step input force of magnitude F applied to the bottom of the rod. Write down an expression for the subsequent displacement of the top end of the rod, paying particular attention to the contribution arising from the rigid body mode. (Hint: let $\omega_n \to 0$ in the step response formula given in the data sheet). Explain in simple physical terms the result obtained for the motion arising from the rigid body mode. [30%]

(d) A satellite is to be carried in the payload bay at the top of Ariane 5. The vehicle is 50 m long, and the speed of sound in the construction material is 5000 m/s. Comment on the vibration environment experienced by the satellite and explain why the designers of the satellite should avoid internal natural frequencies below 100Hz. [20%]

2. A string of length L, tension P, and mass per unit length m is attached to a mass-spring system at the point $x = x_0$, as shown in Fig. 1. The mass-spring system has mass M and natural frequency ω_0 (in the absence of the string).

(a) Write down expressions for the natural frequencies and mode shapes of the string alone (i.e. in the absence of the mass-spring system). [15%]

(b) Write down an expression for the driving point displacement transfer function $H(x_0, x_0, \omega)$ of the string alone at the point $x = x_0$. [15%]

(c) Show that the natural frequencies of the coupled system are given by the roots of the equation

$$H(x_0, x_0, \omega) = M^{-1}\left(\omega^2 - \omega_0^2\right)^{-1}$$ [25%]

(d) Letting ω_{cj} represent the jth natural frequency of the coupled system and ω_j represent the jth natural frequency of the string alone, show by graphical means that ω_{cj} lies closer to the mass-spring natural frequency ω_0 than does ω_j. [25%]

(e) Describe carefully the natural frequencies which are obtained in the limit as $M \to \infty$. [20%]

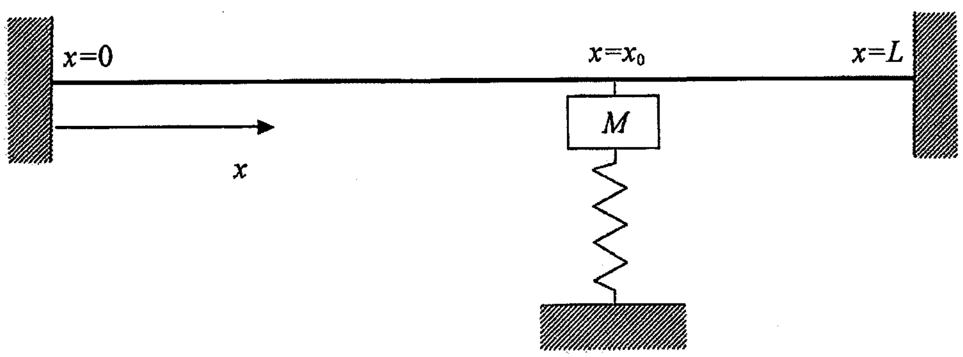

Fig. 1

3. The machinery raft of a ship can be modelled as shown in Fig. 2. A light, rigid, rectangular plate of dimensions $4a \times 4b$ is supported at its corners by four identical springs of stiffness k, resting on a flat, rigid floor. It carries three masses m at the positions shown.

(a) Derive expressions for the potential and kinetic energies, about the position of stable equilibrium, for small vibration of the system described by the three generalized coordinates z (vertical displacement of the centre), θ and ϕ (angles of rotation about the mid-lines of the plate). Hence show that, for coordinate vector $[z\ \theta\ \phi]^T$, the mass and stiffness matrices can be written:

$$[M]=m\begin{bmatrix}3 & 0 & -a\\ 0 & 2b^2 & 0\\ -a & 0 & 3a^2\end{bmatrix},\qquad [K]=k\begin{bmatrix}4 & 0 & 0\\ 0 & 16b^2 & 0\\ 0 & 0 & 16a^2\end{bmatrix}.$$

[20%]

(b) One vibration mode shape can be deduced without calculation. Describe it and give its natural frequency. Find the remaining two vibration modes and their natural frequencies. Sketch these two mode shapes. [60%]

(c) Without further calculations, sketch the amplitude (on a dB scale) of the transfer function for displacement at the point A in response to vertical harmonic force applied at the point B. [20%]

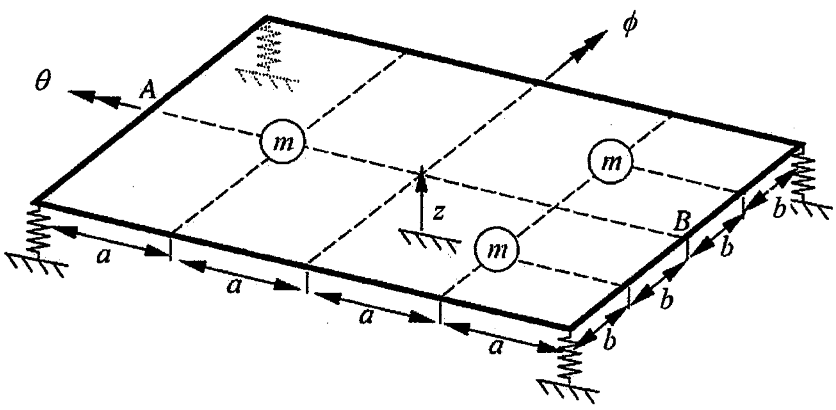

Fig. 2

4. Torsional vibration of an engine is represented by the model shown in Fig. 3. Five discs, each with polar moment of inertia J are connected by shaft sections, three of which have torsional stiffness k and one of which has torsional stiffness S. The assembly is supported by frictionless bearings. The angular positions of the discs are defined by the coordinate vector $[\theta_1 \quad \theta_2 \quad \theta_3 \quad \theta_4 \quad \theta_5]^T$.

(a) Write expressions for the kinetic and potential energies of the system. [10%]

(b) Using the results from (a), write an expression for Rayleigh's quotient. Show that the quotient gives the correct natural frequency for the rigid body mode of vibration. [20%]

(c) For the case where $S = k$, the mode with the smallest non-zero natural frequency has the form $[1 \quad \alpha \quad 0 \quad -\alpha \quad -1]^T$. By minimising Rayleigh's quotient, find the natural frequency and the corresponding mode shape. Comment on any other modes found by this analysis. [30%]

(d) If the stiffness of shaft S is increased by 10%, ($S = 1.1k$), estimate the percentage change in the smallest non-zero natural frequency. [40%]

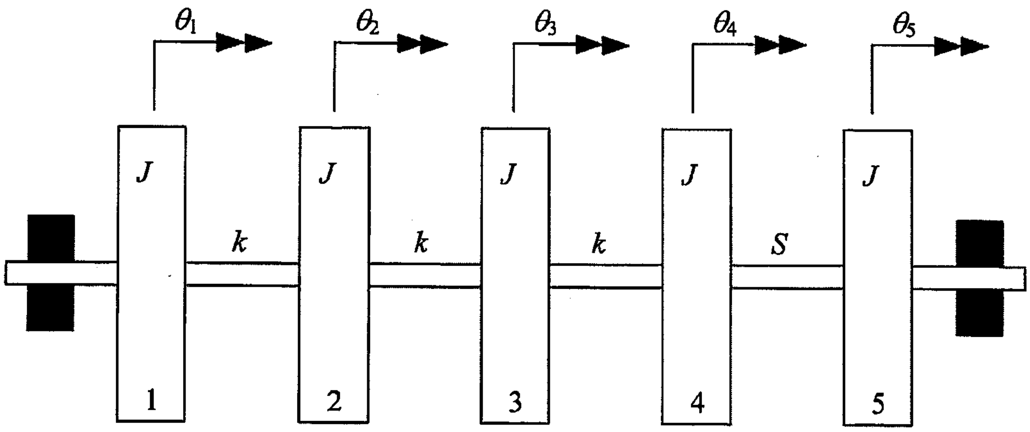

Fig. 3

END OF PAPER

ENGINEERING TRIPOS PART IIA

Module 3C6 Examination, 2011

Answers

1. (b) $u_0(x) = \sqrt{\dfrac{1}{\rho LA}}$; $u_n(x) = \sqrt{\dfrac{2}{\rho LA}}\cos\dfrac{n\pi x}{L}$, $n = 1,2,3...$

(c) $Fh(L,0,t) = \dfrac{t^2}{2}\left(\dfrac{F}{\rho AL}\right) + F\sum_{n=1}^{\infty}\dfrac{u_n(0)u_n(L)}{\omega_n^2}(1-\cos\omega_n t)$ (d) $f_n = 50n$ $Hz.$

2. (a) $\omega_n = \left(\dfrac{n\pi}{L}\right)\sqrt{\dfrac{P}{m}}$; $u_n(x) = \sqrt{\dfrac{2}{mL}}\sin\dfrac{n\pi x}{L}$, $n = 1,2,3...$

(b) $H(x_o, x_o, \omega) = \left(\dfrac{2}{mL}\right)\sum_{n=1}^{\infty}\dfrac{\sin^2(\frac{n\pi x_o}{L})}{\omega_n^2 - \omega^2}$

3. (a)

$$V = \frac{1}{2}k\left\{[z+2a\phi+2b\theta]^2 + [z+2a\phi-2b\theta]^2 + [z-2a\phi+2b\theta]^2 + [z-2a\phi-2b\theta]^2\right\}$$

$$T = \frac{1}{2}m\left\{[\dot{z}+a\dot{\phi}]^2 + [\dot{z}-a\dot{\phi}-b\dot{\theta}]^2 + [\dot{z}-a\dot{\phi}+b\dot{\theta}]^2\right\}$$

(b)

$$\omega^2 = \frac{8k}{m};\quad [z \quad \theta \quad \phi]^T = [0 \quad 1 \quad 0]^T$$

$$\omega^2 = \frac{1.29k}{m};\quad \frac{z}{a\phi} = 9.9;\quad \omega^2 = \frac{6.21k}{m};\quad \frac{z}{a\phi} = -0.42$$

4. (a)

$$V = \frac{1}{2}k\left\{[\theta_2-\theta_1]^2 + [\theta_3-\theta_2]^2 + [\theta_4-\theta_3]^2\right\} + \frac{1}{2}S[\theta_5-\theta_4]^2$$

$$T = \frac{1}{2}J\left\{\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2 + \dot{\theta}_4^2 + \dot{\theta}_5^2\right\}$$

(b) $\dfrac{k}{J}\dfrac{(2\alpha^2 - 2\alpha + 1)}{(1+\alpha^2)}$ (c) $\alpha = -\dfrac{1}{2} \pm \dfrac{\sqrt{5}}{2}$; $\omega = 0.618\sqrt{\dfrac{k}{J}}$, $\omega = 1.618\sqrt{\dfrac{k}{J}}$ (d) 0.686%

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