ENGINEERING TRIPOS PART IIA

Monday 9 May 2011 2.30 to 4.00

Module 3C7

MECHANICS OF SOLIDS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
Special datasheet: Module 3C7 Mechanics of Solids (2 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 A gear wheel is to be shrunk fit onto a long circular shaft resulting in an interfacial pressure p. The shaft and gear are both made of steel with a yield strength of 400MPa. The gear wheel is to be modelled as a disk of thickness 20mm, with outer radius 20mm, in contact with a long shaft of radius 10mm, see Fig. 1.

(a) Assuming plane strain, calculate the reduction of the radius of the shaft due to an interfacial pressure p. [20%]

(b) Assuming plane stress, calculate the expansion of the inner radius of the disk due to an interfacial pressure p. [40%]

(c) Recommend a misfit at the interface that gives a safety factor of 2 against first yield of the components. [20%]

(d) Comment on the appropriateness of the use of plane strain and plane stress for this modelling. [20%]

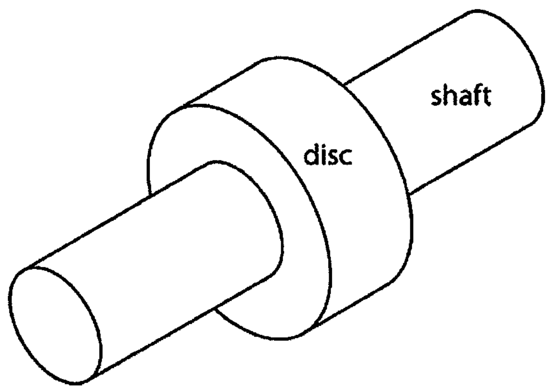

Fig. 1

2 (a) Show that the Airy stress function

$$\phi = -\frac{P}{\pi} r\theta \sin\theta$$

can be used to describe the elastic solution for the stresses due to a line load of magnitude P acting on a half-space, as shown in Fig. 2(a). [40%]

(b) Two line loads, each of magnitude P and separated by a distance $2d$, are now applied to the half-space, as shown in Fig. 2(b). They give rise to in-plane principal stresses $\sigma_1(x,y)$ and $\sigma_2(x,y)$ within the half-space.

(i) Show that $\sigma_1 - \sigma_2 = 0$ at $(x,y) = (0,d)$. [20%]

(ii) Calculate $|\sigma_1 - \sigma_2|$ at $(x,y) = \left(0, d/\sqrt{3}\right)$. [40%]

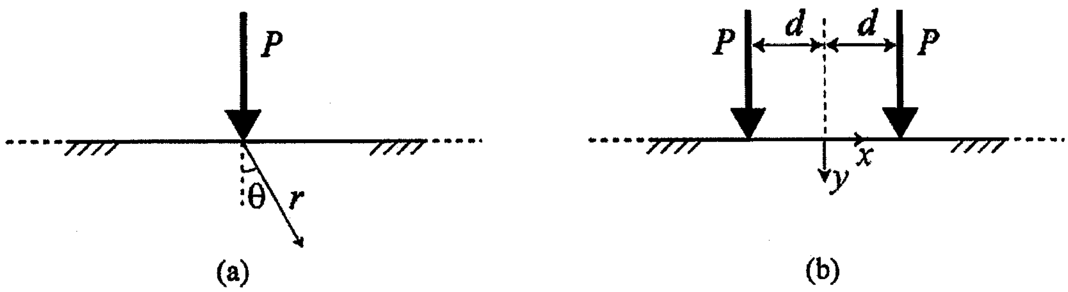

Fig. 2

3 Figure 3 shows the cross-section of a triangular bar under torsion of magnitude T. The bar is made from an elastic solid of shear modulus G.

(a) Consider the Prandtl stress function

$$\psi = -\frac{15\sqrt{3}T}{a^4}\left[\frac{1}{2}(x^2+y^2)-\frac{1}{2a}(x^3-3xy^2)-\frac{2}{27}a^2\right].$$

(i) Without doing calculations, describe how you would show that the stresses defined by ψ are in equilibrium with the applied load. [10%]

(ii) Find the stresses over the cross-section, and show that they satisfy the boundary conditions. [30%]

(iii) Find the twist/unit length of the bar. [20%]

(b) Consider an alternative bar of the same material and cross-sectional area, but which has a circular cross-section. Compare (i) the torsional stiffness, and (ii) the peak shear stress for the two bars. [40%]

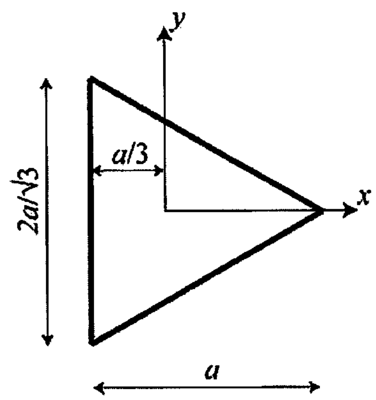

Fig. 3

4 (a) An isotropic elastic ideally-plastic material of yield strength σ_Y and Poisson's ratio $\nu = 1/2$ is loaded in plane strain (ie. $\varepsilon_{33} = 0$).

(i) Write the von Mises yield criterion in terms of the in-plane stress components $(\sigma_{11}, \sigma_{22}, \sigma_{12})$. [20%]

(ii) The material is subjected to a stress state $\sigma_{11} = S, \sigma_{22} = \sigma_{12} = 0$. Find the maximum value of S/σ_Y, if the material yields according to the Tresca yield criterion. [20%]

(b) Find a good upper bound for the full plastic bending moment M (per unit thickness perpendicular to the diagram) for the notched bar as sketched in Fig. 4. The bar is made from a rigid-plastic solid of shear yield strength k. Possible curves of velocity discontinuity are circular arcs across the net section as shown by the dashed lines in Fig. 4. In estimating the minimum upper bound you may assume that the minimum of the function $f(x) \equiv x/\sin^2 x$ is 1.38. [60%]

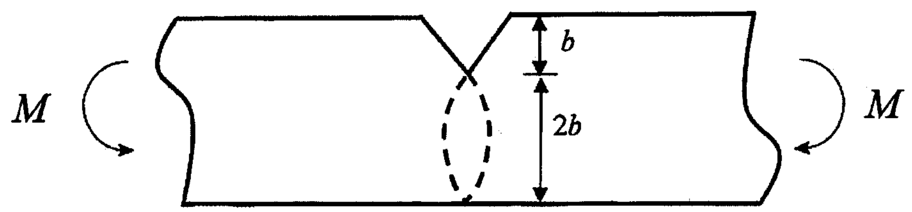

Fig. 4

END OF PAPER

Module 3C7: Mechanics of Solids

ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

	Discs and tubes	Spheres
Equilibrium	$\sigma_{\theta\theta} = \dfrac{d(r\sigma_{rr})}{dr} + \rho\omega^2 r^2$	$\sigma_{\theta\theta} = \dfrac{1}{2r}\dfrac{d(r^2\sigma_{rr})}{dr}$
Lamé's equations (in elasticity)	$\sigma_{rr} = A - \dfrac{B}{r^2} - \dfrac{3+\nu}{8}\rho\omega^2 r^2 - \dfrac{E\alpha}{r^2}\int_c^r rT\,dr$	$\sigma_{rr} = A - \dfrac{B}{r^3}$
	$\sigma_{\theta\theta} = A + \dfrac{B}{r^2} - \dfrac{1+3\nu}{8}\rho\omega^2 r^2 + \dfrac{E\alpha}{r^2}\int_c^r rT\,dr - E\alpha T$	$\sigma_{\theta\theta} = A + \dfrac{B}{2r^3}$

2. Plane stress and plane strain

	Cartesian coordinates	Polar coordinates
Strains	$\varepsilon_{xx} = \dfrac{\partial u}{\partial x}$	$\varepsilon_{rr} = \dfrac{\partial u}{\partial r}$
	$\varepsilon_{yy} = \dfrac{\partial v}{\partial y}$	$\varepsilon_{\theta\theta} = \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$
	$\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$	$\gamma_{r\theta} = \dfrac{\partial v}{\partial r} + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r}$
Compatibility	$\dfrac{\partial^2\gamma_{xy}}{\partial x\partial y} = \dfrac{\partial^2\varepsilon_{xx}}{\partial y^2} + \dfrac{\partial^2\varepsilon_{yy}}{\partial x^2}$	$\dfrac{\partial}{\partial r}\left\{r\dfrac{\partial\gamma_{r\theta}}{\partial\theta}\right\} = \dfrac{\partial}{\partial r}\left\{r^2\dfrac{\partial\varepsilon_{\theta\theta}}{\partial r}\right\} - r\dfrac{\partial\varepsilon_{rr}}{\partial r} + \dfrac{\partial^2\varepsilon_{rr}}{\partial\theta^2}$
or (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}(\sigma_{xx} + \sigma_{yy}) = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right\}(\sigma_{rr} + \sigma_{\theta\theta}) = 0$
Equilibrium	$\dfrac{\partial\sigma_{xx}}{\partial x} + \dfrac{\partial\sigma_{xy}}{\partial y} = 0$	$\dfrac{\partial}{\partial r}(r\sigma_{rr}) + \dfrac{\partial\sigma_{r\theta}}{\partial\theta} - \sigma_{\theta\theta} = 0$
	$\dfrac{\partial\sigma_{yy}}{\partial y} + \dfrac{\partial\sigma_{xy}}{\partial x} = 0$	$\dfrac{\partial\sigma_{\theta\theta}}{\partial\theta} + \dfrac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$
$\nabla^4\phi = 0$ (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}\left\{\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2}\right\} = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right\} \times \left\{\dfrac{\partial^2\phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}\right\} = 0$
Airy Stress Function	$\sigma_{xx} = \dfrac{\partial^2\phi}{\partial y^2}$	$\sigma_{rr} = \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}$
	$\sigma_{yy} = \dfrac{\partial^2\phi}{\partial x^2}$	$\sigma_{\theta\theta} = \dfrac{\partial^2\phi}{\partial r^2}$
	$\sigma_{xy} = -\dfrac{\partial^2\phi}{\partial x\partial y}$	$\sigma_{r\theta} = -\dfrac{\partial}{\partial r}\left\{\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}\right\}$

3. Torsion of prismatic bars

Prandtl stress function: $\sigma_{zx}\ (=\tau_x) = \dfrac{\partial\psi}{\partial y}, \quad \sigma_{zy}\ (=\tau_y) = -\dfrac{\partial\psi}{\partial x}$

Equilibrium: $T = 2\int_A \psi\, dA$

Governing equation for elastic torsion: $\nabla^2\psi = -2G\beta$ where β is the angle of twist per unit length.

4. Total potential energy of a body

$$\Pi = U - W$$

where $U = \frac{1}{2}\int_V \underset{\sim}{\varepsilon}^T [D]\, \underset{\sim}{\varepsilon}\, dV$, $W = \underset{\sim}{P}^{\,T} \underset{\sim}{u}$ and $[D]$ is the elastic stiffness matrix.

5. Principal stresses and stress invariants

Values of the principal stresses, σ_P, can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx}-\sigma_P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy}-\sigma_P & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\sigma_P \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_P.

Expanding: $\sigma_P^3 - I_1\sigma_P^2 + I_2\sigma_P - I_3 = 0$ where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}.$$

6. Equivalent stress and strain

Equivalent stress $\bar{\sigma} = \sqrt{\frac{1}{2}}\ \{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\}^{1/2}$

Equivalent strain increment $d\bar{\varepsilon} = \sqrt{\frac{2}{3}}\ \{d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2\}^{1/2}$

7. Yield criteria and flow rules

Tresca

Material yields when maximum value of $|\sigma_1-\sigma_2|$, $|\sigma_2-\sigma_3|$ or $|\sigma_3-\sigma_1| = Y = 2k$, and then,

if σ_3 is the intermediate stress, $d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

von Mises

Material yields when, $(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{d\varepsilon_1}{\sigma'_1} = \frac{d\varepsilon_2}{\sigma'_2} = \frac{d\varepsilon_3}{\sigma'_3} = \frac{d\varepsilon_1-d\varepsilon_2}{\sigma_1-\sigma_2} = \frac{d\varepsilon_2-d\varepsilon_3}{\sigma_2-\sigma_3} = \frac{d\varepsilon_3-d\varepsilon_1}{\sigma_3-\sigma_1} = \lambda = \frac{3}{2}\frac{d\bar{\varepsilon}}{\bar{\sigma}}.$$

Answers to 3C7: Mechanics of Solids (2010-2011)

1. (a) $-24.7 \times 10^{-15} p \ (m)$; where p is in Pa

   (b) $93.7 \times 10^{-15} p \ (m)$; where p is in Pa

   (c) $8.88 \times 10^{-6} m$

2. (b)(ii) $|\sigma_1 - \sigma_2| = \frac{3P}{2\pi d}$

3. (a)(iii) $\frac{15\sqrt{3}T}{a^4 G}$

   (b)(i) 1.37

   (b)(ii) 1.61

4. (a)(ii) $S / \sigma_Y = 1$

   (b) $2.76 b^2 k$