ENGINEERING TRIPOS PART IIA

Thursday 12 May 2011 2.30 to 4

Module 3D7

FINITE ELEMENT METHODS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: 3D7 Data Sheet (3 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Version: gnw3

1 The governing equations for a particular one-dimensional thermoelasticity problem are

$$-\frac{\partial \sigma}{\partial x} = f$$

$$\frac{\partial T}{\partial t} - k\frac{\partial^2 T}{\partial x^2} + E\beta T_0 \frac{\partial^2 u}{\partial x \partial t} = 0$$

where u is the displacement, T is the temperature and $\sigma = E(du/dx - \beta(T - T_0))$. The terms f, E, T_0, k and β are constant.

(a) If boundary conditions are applied on σ and $k(\partial T/\partial x)$, derive a suitable weak form for this problem. [50%]

(b) If the degrees of freedom for an element are arranged such that $\boldsymbol{a}_e = [u_1, u_2, T_1, T_2]^T$, compute the element mass matrix for this problem for a linear element with length l. [40%]

(c) The semi-discrete global finite element problem for thermoelasticity can be expressed as

$$\boldsymbol{M}\dot{\boldsymbol{a}} + \boldsymbol{K}\boldsymbol{a} = \boldsymbol{b}$$

From this, formulate a fully discrete scheme in terms of $\boldsymbol{a}$ based on the backward Euler method. [10%]

2 (a) Two-dimensional seepage flow through a porous material is considered. The flow rate per unit area in each coordinate direction is given by Darcy's law

$$q_x = -k\frac{\partial h}{\partial x}$$
$$q_y = -k\frac{\partial h}{\partial y}$$

where k is the permeability constant and h is the hydraulic head. Show that the strong form of two-dimensional seepage flow can be expressed as

$$k\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) + s = 0$$

where s is the source of water per unit area. [20%]

(b) On a two dimensional domain Ω with boundary Γ, show that a weak form for seepage flow is

$$k\int_{\Omega}\left(\frac{\partial w_0}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial w_0}{\partial y}\frac{\partial h}{\partial y}\right) d\Omega = \int_{\Omega} w_0 s\, d\Omega - \int_{\Gamma_q} w_0 \bar{q}\, d\Gamma$$

and define Γ_q and $\bar{q}$. [40%]

(c) Consider the three-node triangular element shown in Fig. 1. Compute the component of the element conductance matrix $\boldsymbol{k}_e$ that relates the nodal head at node one (h_1) to the component of the right-hand side vector at node one (f_1). [40%]

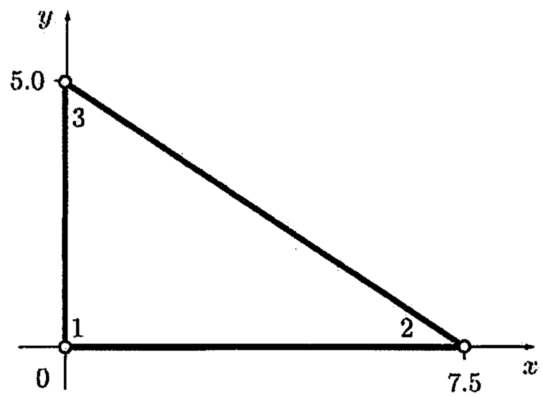


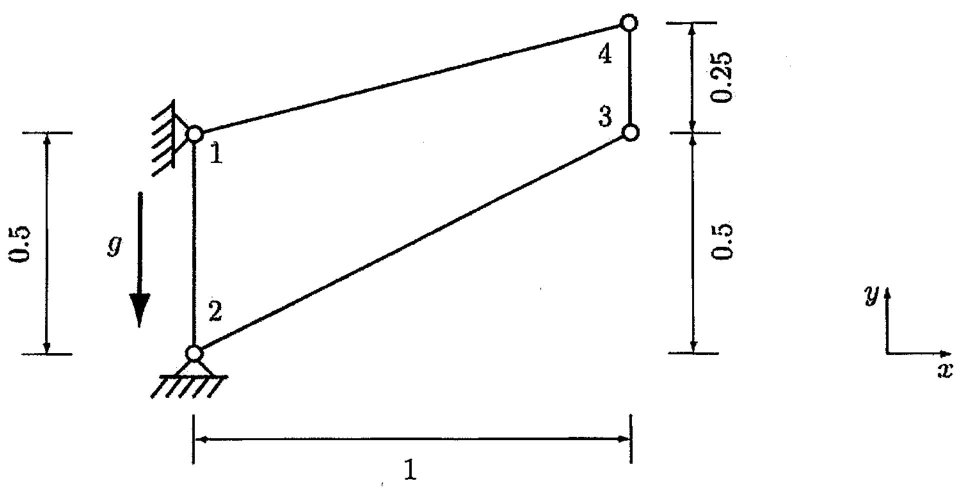
Fig. 1

3 (a) Describe the concept of isoparametric mapping with the help of a one-dimensional two-node element. [20%]

(b) Explain why isoparametric mapping is important for deriving general two and three-dimensional finite elements. [20%]

(c) Consider the linear elastic cantilever structure shown in Fig. 2, which is discretised with one four-noded quadrilateral. The cantilever is subject to gravitational loading of 2000 Nm^{-2} in the direction indicated by g in Fig. 2. Determine the component of the external force vector associated with node 3. [50%]

(d) Briefly comment on the suitability of the mesh shown in Fig. 2 for a finite element analysis. [10%]



All dimensions in metres.

Fig. 2

4 (a) For the equation $\dot{y} = f(y,t)$, the forward Euler scheme is given by

$$y_{n+1} = y_n + \Delta t f(y_n, t_n)$$

and a second-order accurate predictor-corrector scheme involves

$$y_{n+1} = y_n + \Delta t f(y^{\star}_{n+1/2}, t_{n+1/2})$$

where

$$y^{\star}_{n+1/2} = y_n + \frac{\Delta t}{2} f(y_n, t_n)$$

For the model problem $\dot{y} + \lambda y = 0$, where $\lambda > 0$

(i) Compute the amplification factor for the forward Euler scheme and show that the critical time step is

$$\Delta t_{\text{crit}} = \frac{2}{\lambda}$$

[30%]

(ii) Compute the amplification factor for the predictor-corrector scheme and show that the critical time step is the same as for the forward Euler scheme. What is the advantage of the predictor-corrector scheme over the forward Euler scheme? [50%]

(b) Explicit time stepping schemes are commonly applied to elastic wave propagation problems, but less frequently to heat conduction problems. Explain briefly why this is the case. [10%]

(c) For a conditionally stable time stepping scheme applied to elastic wave propagation and for a given element size, would you expect the critical time step to increase or decrease as the polynomial order of the element is increased? Explain why. [10%]

END OF PAPER

Engineering Tripos Part IIA
Module 3D7: Finite Element Methods

Data Sheet

Element relationships

Elasticity

Displacement	$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{a}_e$
Strain	$\boldsymbol{\epsilon} = \boldsymbol{B}\boldsymbol{a}_e$
Stress (2D/3D)	$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\epsilon}$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B}\, dV$
Element force vector (body force only)	$\boldsymbol{f}_e = \int_{V_e} \boldsymbol{N}^T \boldsymbol{f}\, dV$

Heat conduction

Temperature	$T = \boldsymbol{N}\boldsymbol{a}_e$
Temperature gradient	$\nabla T = \boldsymbol{B}\boldsymbol{a}_e$
Element conductance matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B}\, dV$

Beam bending

Displacement	$v = \boldsymbol{N}\boldsymbol{a}_e$
Curvature	$\kappa = \boldsymbol{B}\boldsymbol{a}_e$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T EI \boldsymbol{B}\, dV$

Elasticity matrices

2D plane strain

$$\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

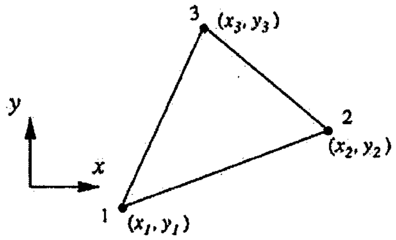
2D plane stress

$$\boldsymbol{D} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

Heat conductivity matrix (2D)

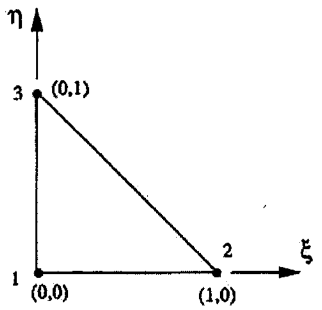
$$\boldsymbol{D} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

Shape functions

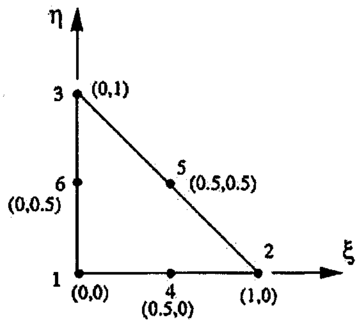


$$N_1 = ((x_2y_3 - x_3y_2) + (y_2 - y_3)\,x + (x_3 - x_2)y)\,/2A$$
$$N_2 = ((x_3y_1 - x_1y_3) + (y_3 - y_1)\,x + (x_1 - x_3)y)\,/2A$$
$$N_3 = ((x_1y_2 - x_2y_1) + (y_1 - y_2)\,x + (x_2 - x_1)y)\,/2A$$

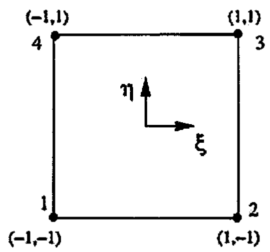
A = area of triangle



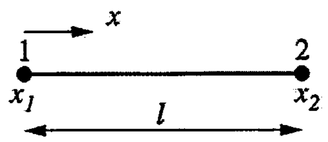
$$N_1 = 1 - \xi - \eta$$
$$N_2 = \xi$$
$$N_3 = \eta$$



$$N_1 = 2(1 - \xi - \eta)^2 - (1 - \xi - \eta)$$
$$N_2 = 2\xi^2 - \xi$$
$$N_3 = 2\eta^2 - \eta$$
$$N_4 = 4\xi(1 - \xi - \eta)$$
$$N_5 = 4\eta\xi$$
$$N_6 = 4\eta(1 - \xi - \eta)$$



$$N_1 = (1 - \xi)(1 - \eta)\,/4$$
$$N_2 = (1 + \xi)(1 - \eta)\,/4$$
$$N_3 = (1 + \xi)(1 + \eta)\,/4$$
$$N_4 = (1 - \xi)(1 + \eta)\,/4$$



Hermitian element

$$N_1 = \frac{-(x-x_2)^2(-l+2(x_1-x))}{l^3}$$

$$M_1 = \frac{(x-x_1)(x-x_2)^2}{l^2}$$

$$N_2 = \frac{(x-x_1)^2(l+2(x_2-x))}{l^3}$$

$$M_2 = \frac{(x-x_1)^2(x-x_2)}{l^2}$$

Gauss integration in one dimension on the domain $(-1,1)$

Using n Gauss integration points, a polynomial of degree $2n-1$ is integrated exactly.

number of points n	location ξ_i	weight w_i
1	0	2
2	$-\frac{1}{\sqrt{3}}$	1
	$\frac{1}{\sqrt{3}}$	1
3	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
	0	$\frac{8}{9}$
	$\sqrt{\frac{3}{5}}$	$\frac{5}{9}$