ENGINEERING TRIPOS PART IIA

Thursday 12 May 2011 9.00 to 10.30

Module 3F3

SIGNAL AND PATTERN PROCESSING

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Version 3

1 (a) The DFT of a sequence $\{x_n\}$ is $\{X_p\}$, and that of a second sequence $\{y_n\}$ is $\{Y_p\}$, for $n = 0, 1, ..., N-1$ and $p = 0, 1, ..., N-1$. We obtain a third DFT, $Z_p = X_p Y_p$, by multiplying X_p and Y_p. Show that the inverse DFT of $\{Z_p\}$ can be expressed in the following form:

$$z_n = \sum_{m=0}^{N-1} y_m x_{\text{mod}(n-m,N)}$$

where mod(P,N) denotes the number P represented in modulo N arithmetic, e.g. $\text{mod}(3,10) = \text{mod}(13,10) = 3$, etc. [40%]

(b) Now, assume that the sequence $\{y_n\}$ contains non-zero values only in its first M elements, i.e.

$$y_n = \{y_0, y_1, \dots, y_{M-1}, 0, 0, \dots, 0\}.$$

Explain why, in this case, the inverse DFT of $\{Z_p\}$ is equal to the standard discrete time convolution of $\{x_n\}$ with $\{y_n\}$, but that this only applies for time indexes $n = M, ..., N-1$, i.e. that:

$$z_n = \sum_{m=0}^{M-1} y_m x_{n-m}, \quad n = M, \dots, N-1.$$

[20%]

(c) (i) Describe how the results in (a) and (b) above can be used to implement FIR filtering with $M = 100$ filter taps on a long sequence of data whose length is $N - M$, where $N = 2^Q > M$, and Q is an integer. Assume that all data prior to time $n = 0$ are zero-valued. [15%]

(ii) Determine the computational load of such a scheme, in terms of real additions and multiplications, in the case where the DFTs are implemented using an appropriate fast algorithm. [10%]

(iii) Determine the length of data N below which the fast DFT-based algorithm will achieve a reduction in total operation count (real multiplications plus additions) compared with a direct time-doman implementation of the $M = 100$ filter. [15%]

You may assume that the DFT of the filter's impulse response has been pre-computed and that N is much greater than M.

2 (a) Explain the terms IIR and FIR in digital filtering. How is the filter's impulse response calculated from the filter coefficients in each case? Describe, with diagrams, how to implement an IIR filter using Direct Form I and Direct Form II structures, indicating any advantages or disadvantages of the two approaches. [30%]

(b) It is proposed to convert an analogue prototype low-pass filter using the substitution formula

$$s \to \frac{s^2 + \omega_1 \omega_2}{s(\omega_2 - \omega_1)},$$

where s is the standard Laplace variable. ω_2 and ω_1 are constants satisfying $\omega_2 > \omega_1$.

Describe the effect of this transformation on the filter's frequency response, explaining with the aid of sketches. Consider in particular what happens to frequencies 0, ω_1, ω_2 and ∞ rad.s^{-1} in the transformed filter. [30%]

(c) A low-pass analogue filter with -3dB frequency of 1 rad.s^{-1} has the following transfer function:

$$H(s) = \frac{1}{s+1}.$$

In a digital audio system it is desired to reduce the effects of bass 'boom' up to 200Hz and percussion noise above 8kHz. The sampling frequency of the system is 44.1 kHz.

Starting with the analogue prototype filter $H(s)$, design a bandpass digital IIR filter according to these criteria, assuming that -3dB attenuation will be adequate at the edges of the filter's pass-band. Sketch the magnitude of its frequency response. [40%]

3 (a) In a data measurement system, some of the data points x_n are found to be heavily corrupted with noise. It is decided to 'clean' the data by performing an interpolation of an anomalous data point at time index n_0 according to the formula

$$\hat{x}_{n_0} = a x_{n_0 - 1} + b x_{n_0 + 1}$$

where x_{n_0-1} and x_{n_0+1} are the data points immediately before and after the corrupted data point at n_0, and there are two constants a and b to be determined.

By considering the above equation to be a digital filter which runs over successive values of n_0 in the signal, determine the frequency response of the interpolation function. In the case $a = b = 0.5$, sketch the frequency response of the filter, paying attention to the DC gain and any 3dB gain points. Is the filter linear phase or not? [30%]

(b) Assume now that $\{x_n\}$ is a wide-sense stationary random process with autocorrelation function $r_{XX}[n]$. Determine the optimal coefficients a and b that will minimize the mean-squared error in the interpolator, i.e. to minimize

$$\varepsilon_{n_0} = E[(x_{n_0} - \hat{x}_{n_0})^2].$$

[30%]

(c) If $r_{XX}[n] = (-0.9)^{|n|}$, determine the mean-squared error corresponding to the optimal interpolator just derived. [20%]

(d) Compare the mean-squared error in part (c) with that of the simple interpolator in part (a), having $a = b = 0.5$. Comment on your result. [20%]

4 Consider a binary classification problem with scalar real-valued observations x, and class labels $y \in \{0, 1\}$. Assume a model with parameters θ where

$$p(y=1|x,\theta) = \frac{1}{1+e^{(-\theta x+1/2)}}$$

(a) Describe the online learning rule for learning the parameter θ assuming the learning algorithm receives one data point at a time. [40%]

(b) Consider a data set $\mathscr{D}$ consisting of three data points: $(x_1 = 0,\ y_1 = 1)$, $(x_2 = -1,\ y_2 = 0)$, and $(x_3 = 1,\ y_3 = 1)$. Compute the likelihood for the parameters θ given this data set $\mathscr{D}$. [30%]

(c) Characterise the solution(s) to the maximum likelihood estimate of θ in part (b) above. Discuss properties of these solution(s), indicating any problems with the result and possible ways of resolving those problems. [30%]

END OF PAPER