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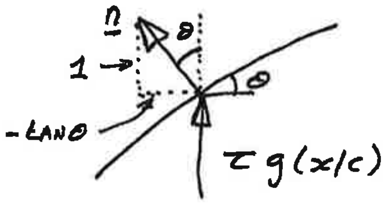
Q1 a) Steady, isentropic, irrotational, linearised [3]

b) Inclination of surface $\tau g(x/c)$ is: $\tan\theta = \tau g'/c$



$\Rightarrow$ NORMAL $\left(-\frac{\tau}{c}g', 1\right)$

Velocity $\left(u_\infty + \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right)$ Flow parallel $\Rightarrow \underline{v}.\underline{n} = 0$

$$\underline{v}.\underline{n} = \left(u_\infty + \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right).\left(-\frac{\tau}{c}g', 1\right) = -\frac{\tau}{c}g' u_\infty \underbrace{-\frac{\tau}{c}g'\frac{\partial\phi}{\partial x}}_{\text{non-linear}} + \frac{\partial\phi}{\partial y} = 0$$

$$\Rightarrow \frac{\partial\phi}{\partial y} - \frac{\tau}{c}g' u_\infty = 0 \quad @ \ y = \tau g$$

$$\text{Now } \left.\frac{\partial\phi}{\partial y}\right|_{y=\tau g} = \left.\frac{\partial\phi}{\partial y}\right|_{y=0} + \underbrace{\tau g \left.\frac{\partial^2\phi}{\partial y^2}\right|_{y=0}}_{\text{neglect.}}$$

BOUNDARY CONDITION $\left.\frac{\partial\phi}{\partial y}\right|_{y=0} = \frac{\tau}{c}g' u_\infty \quad 0<x<c$ [7]

c) put $\tilde{x} = x/c \quad \tilde{y} = \beta y/c$

$$\left.\frac{\partial\phi}{\partial y}\right|_0 = \frac{\beta}{c}\frac{\partial\phi}{\partial\tilde{y}} = \frac{\tau}{c}g'(\tilde{x})u_\infty \Rightarrow \left.\frac{\partial\phi}{\partial\tilde{y}}\right|_0 = \frac{\tau}{\beta}u_\infty g'(\tilde{x})$$

$$\text{put } \tilde{\phi} = \frac{\beta\phi}{\tau u_\infty} \Rightarrow \left.\frac{\partial\tilde{\phi}}{\partial\tilde{y}}\right|_0 = g'(\tilde{x}) \quad 0<\tilde{x}<1$$

$$(1-M_\infty^2)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

$$\Rightarrow \frac{(1-M_\infty^2)}{c^2}\frac{\partial^2\phi}{\partial\tilde{x}^2} + \frac{\beta^2}{c^2}\frac{\partial^2\phi}{\partial\tilde{y}^2} = 0$$

$$\frac{1-M_\infty^2}{\beta^2}\frac{\partial^2\tilde{\phi}}{\partial x^2} + \frac{\partial^2\tilde{\phi}}{\partial y^2} = 0 \qquad \text{put } \beta = \sqrt{1-M_\infty^2}$$

$$\Rightarrow \frac{\partial^2\tilde{\phi}}{\partial\tilde{x}^2} + \frac{\partial^2\tilde{\phi}}{\partial\tilde{y}^2} = 0$$

$$\text{MOM}_x: \ -\frac{1}{\rho}\frac{\partial P}{\partial x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \qquad (v \ll u)$$

$$\text{LINEARISE: } \frac{\partial P}{\partial x} = -\rho_\infty u_\infty \frac{\partial u}{\partial x} = -\rho_\infty u_\infty \frac{\partial^2\phi}{\partial x^2}$$

$$\text{INTEGRATE: } P = -\rho_\infty u_\infty \frac{\partial\phi}{\partial x} + fn(y)$$

$$\text{FAR UPSTREAM } (x=-\infty) \quad \frac{\partial\phi}{\partial x} = 0 \ \& \ P = P_\infty \Rightarrow fn(y) = P_\infty$$

Q1c) cont. $\Rightarrow p - p_\infty = -\rho_\infty u_\infty \dfrac{\partial\phi}{\partial x}$

$$\Rightarrow \frac{p-p_\infty}{\frac{1}{2}\rho_\infty u_\infty^2} = -\frac{2}{u_\infty}\frac{\partial\phi}{\partial x} = -\frac{2}{c\,u_\infty}\frac{\partial\phi}{\partial\tilde{x}} = -\frac{2}{c\,u_\infty}\frac{\tau u_\infty}{\beta}\frac{\partial\tilde{\phi}}{\partial\tilde{x}}$$

$$\beta = \sqrt{1-M_\infty^2} \quad \Rightarrow C_p = \frac{p-p_\infty}{\frac{1}{2}\rho_\infty u_\infty^2} = -2\frac{\partial\tilde{\phi}}{\partial\tilde{x}}\frac{\tau}{c\sqrt{1-M_\infty^2}}$$

put $\underline{\underline{k = -2\dfrac{\partial\tilde{\phi}}{\partial\tilde{x}}}}$ $\Rightarrow$ $\underline{\underline{C_p = \dfrac{\tau k}{c\sqrt{1-M_\infty^2}}}}$ $\boxed{8}$

d) INCOMPRESSIBLE $\quad C_L = \dfrac{L}{\frac{1}{2}\rho u_\infty^2 A}$ $\quad$ Loading $= L/A$.

Sea level $\rho = 1.225\ \text{kg/m}^3$ $\quad C_L = 0.5$ $\quad L/A = 10\ \text{kN/m}^2$

$$u^2 = \frac{2L}{A}\frac{1}{\rho C_L}$$

$$\underline{\underline{u = \sqrt{\frac{2\times10^4}{1.225\times0.5}} = 180.7\ \text{m/s}}} \qquad \left(M = \frac{180.7}{340} = 0.531\right) \quad \boxed{3}$$

e) INCLUDE COMPRESSIBILITY. $\quad C_L \rightarrow \dfrac{C_L}{\sqrt{1-M_\infty^2}}$ $\quad$ ($\tilde{\phi}$ indep. of M)

$$\Rightarrow \frac{C_L}{\sqrt{1-M_\infty^2}} = \frac{(L/A)}{\frac{1}{2}\rho a^2 M_\infty^2} \qquad a\Big|_{\text{SEA LEVEL}} = 340\ \text{m/s}$$

$$\Rightarrow \frac{M_\infty^2}{\sqrt{1-M_\infty^2}} = \frac{(L/A)}{C_L \tfrac{1}{2}\rho a^2} = \frac{10^4}{0.5\times0.5\times1.225\times340^2} = 0.282$$

SOLVE $\dfrac{M_\infty^2}{\sqrt{1-M_\infty^2}} = 0.282 \quad \Rightarrow M^4 + 0.282^2 M^2 - 0.282^2 = 0$

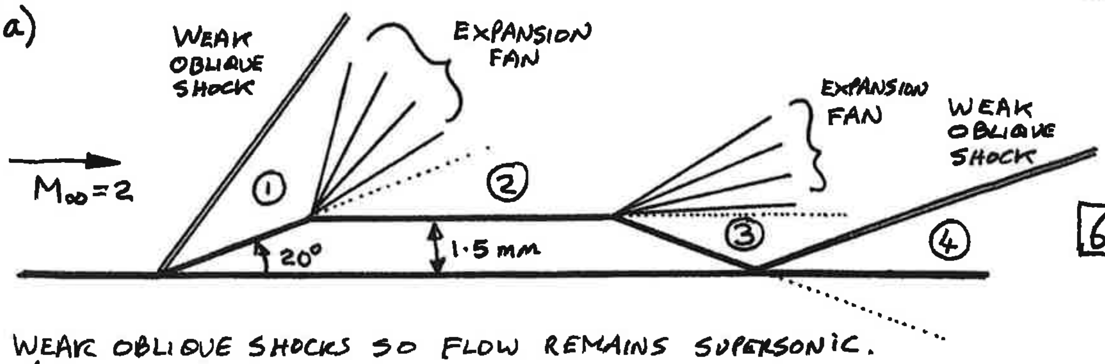
$$M^2 = \frac{-0.282^2 \pm \sqrt{0.282^4 + 4\times0.282^2}}{2}$$

$M = 0.495 \Rightarrow \underline{\underline{V_{MAX} = 0.495\times340 = 168.3\ \text{m/s}}}$ $\boxed{3}$

Examiner's Comments:

Although standard bookwork very few candidates attempted this question. Possibly the lack of intermediate expressions put off candidates.

Q2 a)



$\boxed{6}$

WEAK OBLIQUE SHOCKS SO FLOW REMAINS SUPERSONIC.
(NOTE: OBLIQUE SHOCKS & FANS ARE DRAWN AT APROX. CORRECT ANGLES IN THIS DIAGRAM BASED ON LATER CALCULATIONS.)

b) USING TABLES ($\gamma = 1.4$)

$M_\infty = 2.0 \Rightarrow \theta_\infty = 0^\circ \qquad p_\infty / p_{0\infty} = 0.1278$

WEAK OBLIQUE ($M=2, \theta=20^\circ$)

$\underline{M_1 = 1.2102} \quad \frac{p_1}{p_\infty} = 2.8429 \quad \frac{p_{01}}{p_{0\infty}} = 0.89291 \quad \theta_1 = 20^\circ \quad \nu_1 = 3.815^\circ$

EXPANSION FAN 1 → 2 $\quad \nu + \theta =$ CONSTANT ACROSS +ve μ CHICS.

$\theta_2 = 0^\circ \Rightarrow \nu_2 + \theta_2 = \nu_1 + \theta_1$

$\nu_2 = (3.815^\circ + 20^\circ) - 0^\circ = 23.815^\circ \Rightarrow \underline{M_2 = 1.9080}$

$p_{02} = p_{01}$

EXPANSION FAN 2 → 3

$\theta_3 = -20^\circ \qquad \nu_3 + \theta_3 = \nu_2 + \theta_2$

$p_{03} = p_{02} \qquad \nu_3 = (23.815^\circ) - (-20^\circ) = 43.815^\circ$

$\Rightarrow \underline{M_3 = 2.7089} \qquad p_3/p_{03} = 0.04238$

[CHECK: For $M_3 = 2.7089$ WEAK OBLIQUE SHOCK WITH $\theta = 20^\circ$ IS POSSIBLE SO DO NOT EXPECT DOWNSTREAM SEPARATION]

$\underline{p_1 = 2.8429\, p_\infty}$

$p_3 = 0.04238\, p_{03} = 0.04238\, p_{01} = 0.04238 \times 0.89291\, p_{0\infty}$
$= 0.04238 \times 0.8929 \times (p_\infty / 0.1278)$
$\underline{p_3 = 0.2961\, p_\infty}$

ONLY p_1 AND p_3 CONTRIBUTE TO DRAG FORCE.

$$\frac{\text{DRAG}}{\text{LENGTH}} = \frac{\Delta p \times A}{\text{LENGTH}} = \frac{(2.8429 - 0.2961)\, p_\infty \times 1.5 \times 10^{-3}}{1.0}$$

$$\underline{\underline{= 3.8202 \times 10^{-3}\, p_\infty \quad (\text{N/m})}}$$

$\boxed{12}$

Q2 cont.

c) ALTITUDE = 15000 m $\dfrac{T_\infty}{T_{SL}} = 0.7519$ $\dfrac{P_\infty}{P_{SL}} = 0.1195$

$T_{SL} = 288.15\text{ K} \Rightarrow T_\infty = 216.7\text{ K}$

$P_{SL} = 101325\text{ Pa} \Rightarrow P_\infty = 12.108\text{ kPa}$

EFFECTIVE LENGTH (1000 RIVITS) $= 1000 \times 6\times10^{-3} = 6\text{ m}$

DRAG $= 3.8202\times10^{-3} \times 12108 \times 6 = \underline{277.5\text{ N}}$

$T_\infty = 216.7\text{ K} \Rightarrow a_\infty = \sqrt{\gamma R T_\infty} = \sqrt{1.4\times287\times216.7} = 295.1\text{ m/s}$

$M_\infty = 2.0 \Rightarrow V_\infty = 2\times295.1 = \underline{590.2\text{ m/s}}$

POWER = FORCE × VELOCITY
= 277.5 × 590.2
= 163.8 kW

$\boxed{6}$

EXAMINER'S COMMENTS:

A very popular question which was generally very well done. Common mistakes were not specifying that the shocks were "WEAK OBLIQUE SHOCK" in (a) or by not spotting that the drag force depended only on P_1 and P_3 along with bump height.

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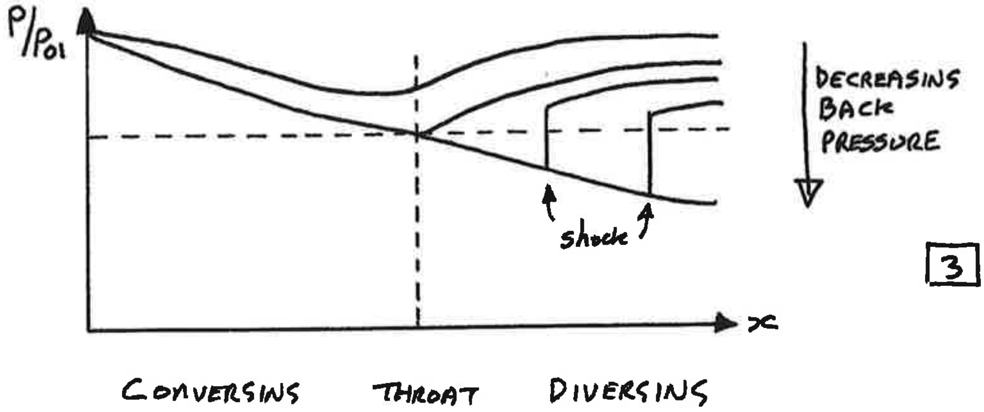
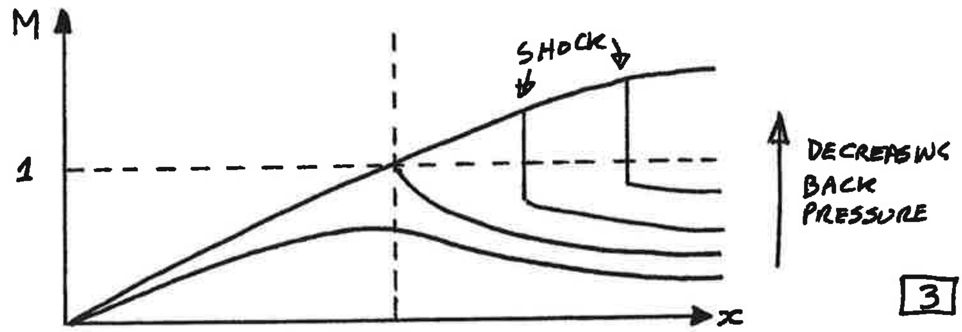
(5)/(16)

(Q3) a)





Key points: High back pressure : Subsonic flow (isentropic)
choking at throat (M=1).
Supersonic expansion, shock, subsonic diffusion.
Low back pressure : Subsonic, M=1, Supersonic (isentropic).

Examiner's comment: most candidates drew both diagrams (as requested) and indicated most features.

b) i) ISA TABLES, 12 km $\quad P/P_{SL} = 0.1915 \quad P_{SL} = 1.01325$ bar

$\Rightarrow \underline{P_{AMB} = 19404 \text{ Pa}}$

ISENTROPIC, $P_{01} = 1.8$ bar $\quad \dfrac{P_e}{P_{01}} = \dfrac{19404}{1.8\times10^5} = 0.1078$

TABLES $\Rightarrow \underline{M_e = 2.11}$

AT THROAT (M=1) $\quad \dfrac{\dot m\sqrt{c_pT_0}}{A^*P_0} = 1.281$ $\qquad$ AT EXIT (M=2.11) $\quad \dfrac{\dot m\sqrt{c_pT_0}}{A_eP_0} = 0.6914$

$\Rightarrow \dfrac{A_e}{A^*} = \dfrac{1.281}{0.6914} = 1.853$ $\qquad A^* = \frac{\pi}{4}(0.2)^2 = 0.0314 \text{ m}^2$

$\Rightarrow A_e = 1.853\times0.0314 = \underline{\underline{0.0582 \text{ m}^2}}$ $\boxed{5}$

Examiner's Comment: Generally well done.

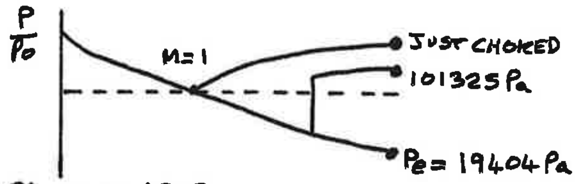
Q3) b) ii) $M_e = 2.11 \Rightarrow \dfrac{F}{\dot{m}\sqrt{c_pT_0}} = 1.1263$

$$F = \frac{F}{\dot{m}\sqrt{c_pT_0}} \times \frac{\dot{m}\sqrt{c_pT_0}}{A^* P_0} \times A^* P_0$$
$$= 1.1263 \times 1.281 \times 0.0314 \times 1.8\times10^5$$
$$\underline{= 8155\ N} \quad (\text{IMPULSE})$$

THRUST $= F - A_eP_e = 8155 - 0.0582 \times 19404 = \underline{\underline{7026\ N}}$ $\boxed{4}$

Examiner's Comment: Most candidates remembered to subtract A_eP_e. (Some used $F - A_eP_e = A_e(\rho u_e^2 + P_e) - A_eP_e = \dot{m}u_e$ since flow isentropic)

c) i) Need to show that flow is choked and non-isentropic.



Need to find P_{exit} for just choked.

$$\frac{\dot{m}\sqrt{c_pT_{01}}}{A_e P_0} = \frac{A^*}{A_e}\frac{\dot{m}\sqrt{c_pT_{01}}}{A^* P_0} = \frac{1.281}{1.853} = 0.6913$$

SUBSONIC $M_e = 0.334$ (SUPERSONIC $M_e = 2.11$)

$\Rightarrow \dfrac{P_{e\,\text{JUST CHOKED}}}{P_{01}} = 0.9257 \Rightarrow P_{e\,\text{JUST CHOKED}} = \underline{166622\ Pa}$

SINCE REQUIRE P_{EXIT} $19404\ Pa < 101325 < 166622$

<u>THERE MUST BE A SHOCK IN THE DIVERGENT SECTION.</u> $\boxed{3}$

Examiner's Comment: Most candidates incorrectly assumed that since $M_{exit} < 1$ there must be a shock! (The flow could be entirely subsonic!).

c) ii) When $P_{exit} = 101325\ Pa$ AND FLOW KNOWN TO BE CHOKED!

$$\frac{\dot{m}\sqrt{c_pT_0}}{A_e P_{exit}} = \frac{\dot{m}\sqrt{c_pT_0}}{A^* P_0} \times \frac{A^*}{A_e} \times \frac{P_0}{P_{exit}} = 1.281 \times \frac{1}{1.853} \times \frac{1.8\times10^5}{101325} = 1.228$$

$\Rightarrow \underline{M_{EXIT} = 0.54\ (\text{SUBSONIC})} \Rightarrow \dfrac{\dot{m}\sqrt{c_pT_0}}{A_e P_{0EXIT}} = 1.0084$

$$\frac{P_{0EXIT}}{P_0} = \frac{\dot{m}\sqrt{c_pT_{01}}}{A^* P_0} \times \frac{A_e P_{0EXIT}}{\dot{m}\sqrt{c_pT_0}} \times \frac{A^*}{A_e} = \frac{1.281}{1.0084} \times \frac{1}{1.853} = \underline{0.6856}$$
(SHOCK LOSS)

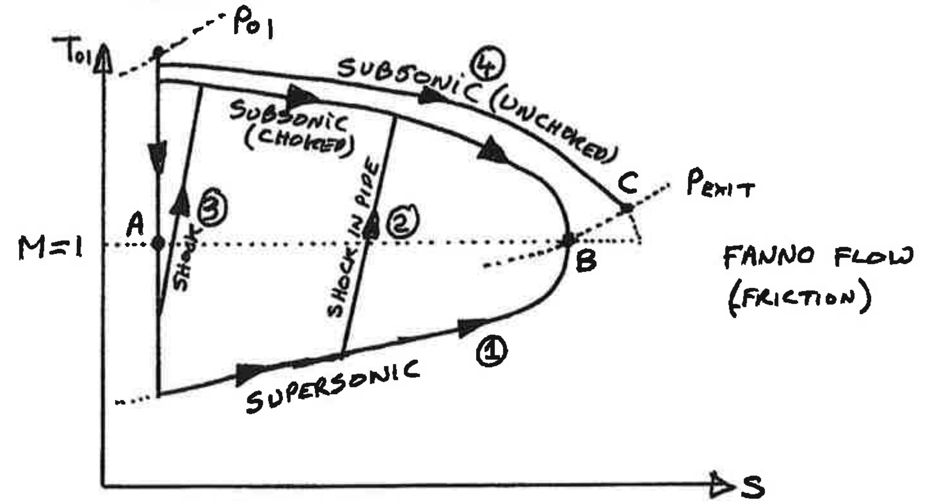
$\gamma = 1.4$, SHOCK TABLES : $\dfrac{P_{0EXIT}}{P_0} = 0.6856 \Rightarrow \underline{M = 2.075}$ $\boxed{3}$
(AHEAD OF SHOCK)

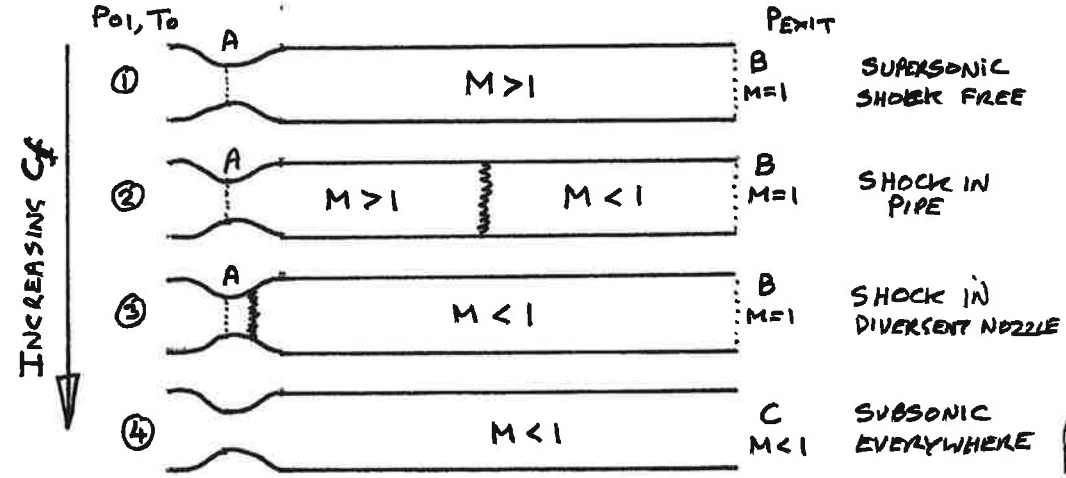
c) iii) $F = \dfrac{F}{\dot{m}\sqrt{c_pT_0}} \times \dfrac{\dot{m}\sqrt{c_pT_0}}{A_e P_{exit}} \times A_e P_{exit} = 1.1452 \times 1.228 \times 0.0582 \times 101325 = 8293\ Pa$

THRUST $= F - A_e P_{exit} = 8293 - 0.0582 \times 101325 = \underline{\underline{2396\ N}}$ $\boxed{3}$

Examiner's Comment: Many candidates got completely lost in trying to determine the shock strength.

Q4 a)





$\boxed{8}$

Examiner's Comment: Generally well understood Fanno flow.

b) i) M=1 AT PIPE EXIT $\Rightarrow \dfrac{\dot{m}\sqrt{c_p T_0}}{A\,P_{EXIT}} = 2.4249$ $\qquad P_{EXIT} = 1$ bar

PIPE INLET $\dfrac{\dot{m}\sqrt{c_p T_0}}{A\,P_{01}} = \dfrac{\dot{m}\sqrt{c_p T_0}}{A\,P_{EXIT}} \times \dfrac{P_{EXIT}}{P_{01}} = \dfrac{2.4249 \times 1.0}{3.14} = 0.7723$

SUBSONIC $\Rightarrow \underline{\underline{M_{INLET} = 0.3800}}$ $\qquad \left.\dfrac{4C_f L}{D}\right|_{INLET} = 2.7054$ $\qquad \underline{\underline{C_f = 0.0090}}$

SUPERSONIC $\Rightarrow \underline{\underline{M_{INLET} = 1.9792}}$ $\qquad \left.\dfrac{4C_f L}{D}\right|_{INLET} = 0.2988$ $\qquad \underline{\underline{C_f = 0.0010}}$

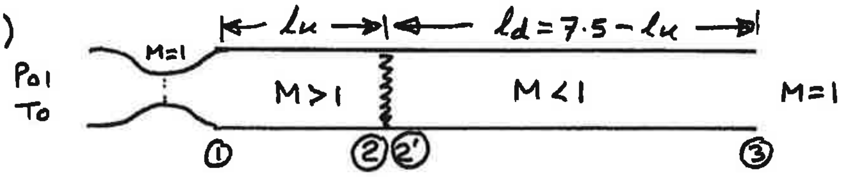
Examiner's Comment: Done well by most. $\boxed{4}$

b) ii) Subsonic $\Rightarrow$ low velocity $\Rightarrow$ high C_f to produce loss.
Supersonic $\Rightarrow$ high velocity $\Rightarrow$ low C_f to produce loss. $\boxed{2}$

Examiner's comment: Most candidates had a rough idea that $\tau_w = \frac{1}{2}\rho v^2 C_f \propto \dot{m}\, v\, C_f$

Q4 cont)

c)i)



At 1 $M_{INLET} = 1.9792$ $\quad \left.\frac{4C_fL}{D}\right|_1 = 0.2988$

SUPERSONIC 1 → 2 $\quad \left.\frac{4C_fL}{D}\right|_2 = \left.\frac{4C_fL}{D}\right|_1 - \frac{4C_f l_u}{D}$

SUBSONIC 2' → 3 $\quad \left.\frac{4C_fL}{D}\right|_3 = \left.\frac{4C_fL}{D}\right|_{2'} - \frac{4C_f l_d}{D}$

$C_f = 0.0015$, $D = 0.1$ $\Rightarrow \frac{4C_f}{D} = 0.06$, REQUIRE $\left.\frac{4C_fL}{D}\right|_3 = 0$

l_u	$\left.\frac{4C_fL}{D}\right|_2$	M_2	$M_{2'}$	$\left.\frac{4C_fL}{D}\right|_{2'}$	l_d	$\left.\frac{4C_fL}{D}\right|_3$
1m	0.2388	1.7909	0.6186	0.4222	6.5m	0.0322
2m	0.1788	1.6180	0.6631	0.2902	5.5m	-0.0398

Since $\left.\frac{4C_fL}{D}\right|_3$ changes sign, require solution has $1 < l_u < 2$. $\boxed{6}$

Examiner's Comment: Few candidates really knew how to determine the shock was between the specified l_u=1&2.

c)ii) REQUIRE $\left.\frac{4C_fL}{D}\right|_3 = 0$ $\quad \Rightarrow$ $l_u \approx 1.45\,m$, $M_2 \approx 1.71$ $\boxed{2}$

c)iii) Since the nozzle remains choked for both the "as manufactured" and "corroded" condition the mass flow remains fixed.

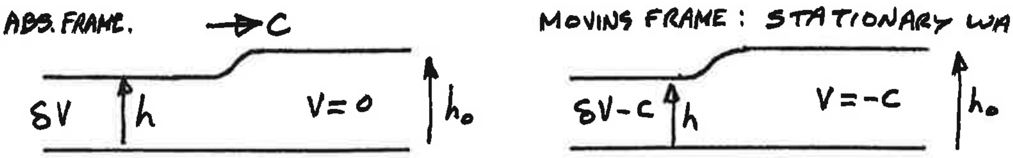
Similarly, since the exit pressure and the reservoir T_0, pressures are unchanged, the overall ΔP is the same.

When the corrosion causes the C_f to increase the shock is introduced so that downstream the velocity is lower, and hence the wall shear stress is reduced downstream of the shock. Thus, overall the pressure drop can remain constant.

$\boxed{2}$

Examiner's Comments: Very few candidates could spot why the mass flow and ΔP were unchanged even when C_f has increased.

Q5 a) ABS. FRAME. MOVING FRAME : STATIONARY WAVE



ENERGY EQN : $gh_o = gh + \frac{1}{2}V^2$

$\Rightarrow \quad 0 = g\,dh + V\delta V$

BUT $V=-c \Rightarrow \quad g\,dh = -V\,dV = c\,dV$

$c = \sqrt{gh} \Rightarrow \quad 2c\,dc = g\,dh$

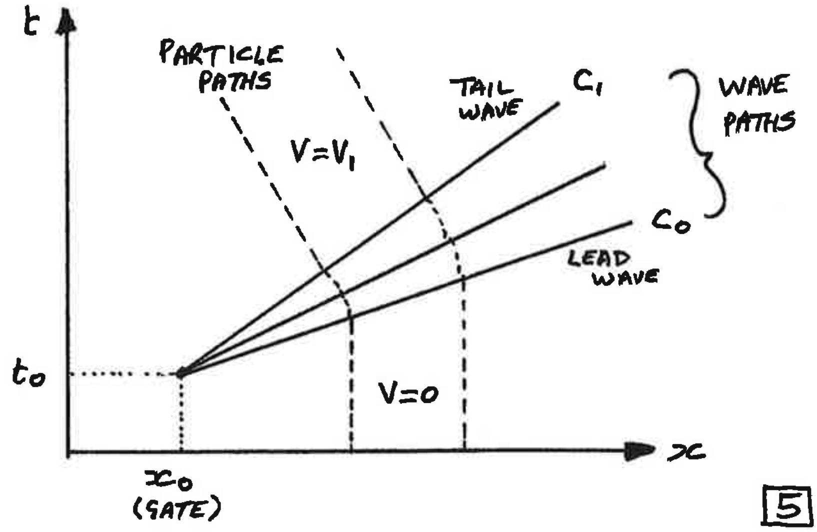
COMBINE $\Rightarrow \quad 2c\,dc = c\,dV$

$\Rightarrow \quad d(V-2c) = 0$

$V - 2c$ = CONSTANT ACROSS RIGHT-RUN WAVE $\boxed{7}$

Examiner's comment: Generally well done but many candidates tried to do this in the absolute frame.

b) SERIES OF INFINITESIMAL WAVES.

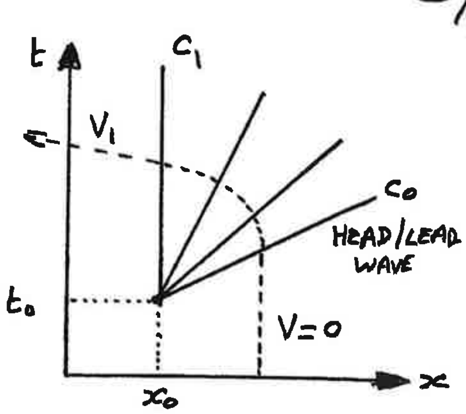


$\boxed{5}$

Examiner's Comments: Most candidates were able to draw a reasonable version of the above.

Q5 cont)

c) LIMITING CASE (CRITICAL CONDITION)
FINAL WAVE $V_1 = -C_1$ SO IS STATIONARY IN ABSOLUTE FRAME.



$$V_0 - 2C_0 = V_1 - 2C_1$$

$V_0 = 0$, $V_1 = -C$ $\quad c = \sqrt{gh}$

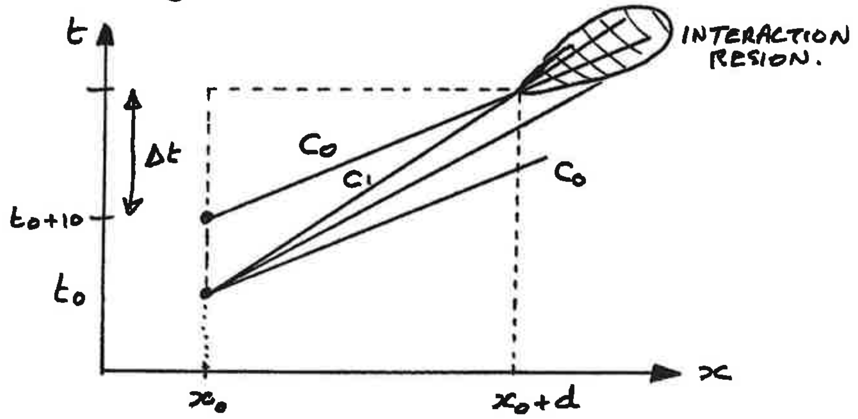
$$\Rightarrow -2C_0 = -3C_1$$

$$\frac{C_0}{C_1} = \frac{3}{2} \qquad \frac{h_0}{h_1} = \left(\frac{3}{2}\right)^2 = \frac{9}{4} \qquad \frac{h_1}{h_0} = \frac{4}{9} = 0.444$$

THIS IS NOT A SMALL CHANGE IN HEIGHT. $\boxed{5}$

Examiner's comments: Some candidates had no idea about the limiting case.

d)



INITIAL DEPTH $h_0 = 0.3$m , DROP ON OPENING = 0.05m $\Rightarrow h_1 = 0.25$m

$\frac{h_0}{h_1} = \frac{0.3}{0.25} = 1.2 \Rightarrow$ FLOW IS SUBCRITICAL.

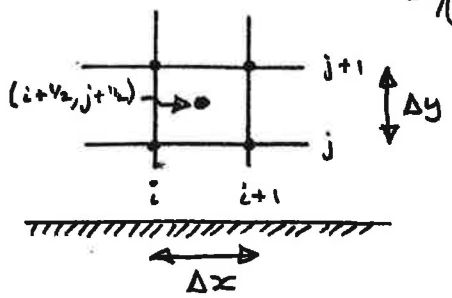
$g = 9.81 \Rightarrow C_0 = \sqrt{gh_0} = 1.716$ m/s
$C_1 = \sqrt{gh_1} = 1.566$ m/s

$$d = C_0 \Delta t = C_1(\Delta t + 10) \qquad \Delta t = \frac{10C_1}{C_0 - C_1} = \underline{104.4s}$$

$$\Rightarrow \underline{d = 1.716 \times 104.4 = 179\text{m}}$$ $\boxed{7}$

Examiner's comments: Many candidates did not know how to draw the x–t diagram or solve the simple geometry.

Q6 a) $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$



DISCRETISE $x = i\,\Delta x$

$y = j\,\Delta y$

$$\left(\frac{u_j^{i+1} - u_j^i}{\Delta x} + \frac{u_{j+1}^{i+1} - u_{j+1}^i}{\Delta x}\right)$$

$$+\left(\frac{v_{j+1}^i - v_j^i}{\Delta y} + \frac{v_{j+1}^{i+1} - v_j^{i+1}}{\Delta y}\right) = 0$$

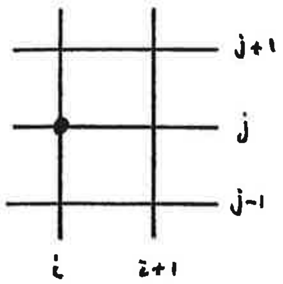
SECOND ORDER ABOUT CELL CENTRE $(i+½, j+½)$

A less suitable alternative is to centre about (i, j) to give $\left(\dfrac{u_j^{i+1} - u_j^{i-1}}{2\Delta x} + \dfrac{v_{j+1}^i - v_{j-1}^i}{2\Delta y}\right) = 0$

This is still second order but is not as closely related to mass conservation evaluated over a cell. $\boxed{6}$

Examiner's comments: Few candidates really understood about the cell centre $(i+½, j+½)$ evaluation.

b) MOMENTUM EQUATION:



$u_j^i\left[\dfrac{u_j^{i+1} - u_j^i}{\Delta x}\right]$ FIRST ORDER IN Δx } could be made 2nd order.

$+ v_j^i\left[\dfrac{u_{j+1}^i - u_{j-1}^i}{\Delta y}\right]$ SECOND ORDER IN Δy

$= -\dfrac{1}{\rho}\left[\dfrac{P_j^{i+1} - P_j^i}{\Delta x}\right]$ FIRST ORDER IN Δx } COULD BE MADE SECOND ORDER USING P_j^{i-1}

$+ \nu\left[\dfrac{u_{j+1}^i - 2u_j^i + u_{j-1}^i}{\Delta y^2}\right]$ SECOND ORDER IN Δy.

Note: Improving the pressure gradient term to second order is usually not a problem as, for most problems, the pressure field is known. $\boxed{8}$

Examiner's comments: Most candidates could get most of the above terms approximately correct.

Q6 cont)

c) METHOD: REARRANGE DISCRETE MOMENTUM EQUATION:

$$u_j^{i+1} = u_j^i - \frac{v_j^i \Delta x}{u_j^i}\left[\frac{u_{j+1}^i - u_{j-1}^i}{2\Delta y}\right] - \underbrace{\frac{p_j^{i+1} - p_j^i}{\rho u_j^i}}_{\text{KNOWN}} + \frac{\nu \Delta x}{u_j^i \Delta y^2}\left[u_{j+1}^i - 2u_j^i + u_{j-1}^i\right]$$

The above allows evaluation of u_j^{i+1} from the data at $x = x_i$. Given the boundary condition $u_1^{i+1} = 0$ (NO SLIP) the above can be marched in the y-direction to determine all u_j^{i+1} values. $\boxed{5}$

REARRANGE CONTINUITY EQUATION:

$$v_{j+1}^{i+1} = v_j^{i+1} - \frac{\Delta y}{\Delta x}\left[u_j^{i+1} - u_j^i + u_{j+1}^{i+1} - u_{j+1}^i\right] - \left[v_{j+1}^i - v_j^i\right]$$

Given the boundary condition $v_1^{i+1} = 0$ (impermeable) and the values for u_{j+1}^{i+1} calculated above, the continuity equation can be marched in the y-direction to determine all the v_{j+1}^{i+1} values.

The above two steps are then repeated at the next x-location. $\boxed{5}$

Examiner's comments: No candidates were completely able to explain the above two-part solution method. However, many were able to explain "general marching methods".

Q7 a) i) TAYLOR SERIES IN SPACE:

$$\theta_{i+1}^n = \theta_i^n + \Delta x \frac{\partial\theta}{\partial x}\Big|_i^n + \frac{\Delta x^2}{2!}\frac{\partial^2\theta}{\partial x^2}\Big|_i^n + \frac{\Delta x^3}{3!}\frac{\partial^3\theta}{\partial x^3}\Big|_i^n + \frac{\Delta x^4}{4!}\frac{\partial^4\theta}{\partial x^4}\Big|_i^n + O(\Delta x^5)$$

$$\theta_{i-1}^n = \theta_i^n - \Delta x \frac{\partial\theta}{\partial x}\Big|_i^n + \frac{\Delta x^2}{2!}\frac{\partial^2\theta}{\partial x^2}\Big|_i^n - \frac{\Delta x^3}{3!}\frac{\partial^3\theta}{\partial x^3}\Big|_i^n + \frac{\Delta x^4}{4!}\frac{\partial^4\theta}{\partial x^4}\Big|_i^n + O(\Delta x^5)$$

$$\Rightarrow \frac{\partial^2\theta}{\partial x^2}\Big|_i^n = \frac{\theta_{i+1}^n - 2\theta_i^n + \theta_{i-1}^n}{\Delta x^2} \underbrace{- \frac{\Delta x^2}{12}\frac{\partial^4\theta}{\partial x^4}\Big|_i^n} + O(\Delta x^3)$$

$\Rightarrow$ Second order in space

TAYLOR SERIES IN TIME:

$$\theta_i^{n+1} = \theta_i^n + \Delta t \frac{\partial\theta}{\partial t}\Big|_i^n + \frac{\Delta t^2}{2!}\frac{\partial^2\theta}{\partial t^2}\Big|_i^n + O(\Delta t^3)$$

$$\Rightarrow \frac{\partial\theta}{\partial t}\Big|_i^n = \frac{\theta_i^{n+1} - \theta_i^n}{\Delta t} \underbrace{- \frac{\Delta t}{2}\frac{\partial^2\theta}{\partial t^2}\Big|_i^n} + O(\Delta t^2)$$

$\Rightarrow$ First order in time $\boxed{7}$

a) ii) DISCRETE PERTURBATION LET: $\theta_i^n = \varepsilon$, $\theta_{i+1}^n = \theta_{i-1}^n = -\varepsilon$

$$\theta_i^{n+1} = \varepsilon + \beta(-\varepsilon - 2\varepsilon - \varepsilon) \qquad \beta = \frac{\alpha\Delta t}{(\Delta x)^2}$$

$$\frac{\theta_i^{n+1}}{\varepsilon} = 1 - 4\beta$$

STABILITY $\left|\frac{\theta_i^{n+1}}{\varepsilon}\right| < 1 \Rightarrow -1 < 1 - 4\beta < 1$

$$0 < \beta < 1/2$$ $\boxed{5}$

Examiner's Comments: Part (a) was done very well.

b) i) $T\Delta S = \Delta h_0 - \frac{1}{\rho}\Delta P_0 \Rightarrow \Delta h_0^{ISEN} = \frac{\Delta P_0}{\rho}$ AND $\frac{1}{\rho}\Delta P_0^{ISEN} = \Delta h_0$

$$\eta = \frac{\Delta h_0^{ISEN}}{\Delta h_0} = \frac{1}{\rho}\frac{\Delta P_0}{\Delta h_0} = \frac{\Delta P_0}{\Delta P_0^{ISEN}}.$$

$$\frac{\Delta P_0^{ISEN} - \Delta P_0}{\rho \Omega^2 D^2} = \frac{\Delta P_0^{ISEN}}{\rho\Omega^2 D^2}\left(1 - \frac{\Delta P_0}{\Delta P_0^{ISEN}}\right) = \frac{\Delta P_0^{ISEN}}{\rho\Omega^2 D^2}(1-\eta) \propto Re^{-0.2}$$

GIVEN $\frac{\Delta P_0^{ISEN}}{\rho\Omega^2 D^2} = fn(\phi) \Rightarrow (1-\eta) \propto Re^{-0.2}$ FOR $\phi = $ const. $\boxed{4}$

Examiner's Comment: Many candidates could not get to $\eta = \Delta P_0 / \Delta P_0^{ISEN}$ for a pump!

Q7 cont.) NON-DIMENSIONAL GROUPS:

b) ii) $\dfrac{\Delta P_0}{\rho \Omega^2 D^2} = fn(\phi, Re)$ $\quad \dfrac{POWER}{\rho \Omega^3 D^5} = fn(\phi)$ $\quad \dfrac{\dot{Q}}{\Omega D^3} = fn(\phi)$

Now $\Delta P_0 = \rho g(\text{HEAD})$ $\quad$ $\text{Power} = \dot{m}\,\Delta h_0 = \dot{Q}\rho\,\Delta h_0 = \dot{Q}\,\Delta P_0^{ISEN}$

So $\underline{\Delta P_0 = \eta\,\Delta P_0^{ISEN} = \dfrac{\eta\,\text{Power}}{\dot{Q}}}$ $\quad \left(\& \ \dfrac{\Delta P_0}{\rho \Omega^2 D^2} = \eta\, \dfrac{\text{Power}/\rho\Omega^3 D^5}{\dot{Q}/\Omega D^3} \right)$

DATA:	FULLSIZE	TEST
DIAMETER (D)	3.8 m	0.4 m
SPEED (Ω)	500 rpm	3000 rpm
VOL. FLOW RATE ($\dot{Q}$)	50 m³/s	(CALC ⇒ 0.350 m³/s)
η EFFICIENCY.	90%	(CALC ⇒ 82.8%)
HEAD	500 m	(CALC ⇒ 183.5 m)
ΔP_0	(CALC ⇒ 4.91 MPa)	(CALC ⇒ 1.8 MPa)
POWER	(CALC ⇒ 272.8 MW)	(CALC ⇒ 0.761 MW)

$$Re = \frac{\rho U D}{\mu} \propto \frac{\rho D^2 \Omega}{\mu} \qquad \frac{Re|_{TEST}}{Re|_{FULL}} = \left(\frac{D_{TEST}}{D_{FULL}}\right)^2 \left(\frac{\Omega_{TEST}}{\Omega_{FULL}}\right) = \left(\frac{0.4}{3.8}\right)^2 \left(\frac{3000}{500}\right) = 0.0665$$

$$\frac{(1-\eta)_{TEST}}{(1-\eta)_{FULL}} = \left(\frac{Re|_{TEST}}{Re|_{FULL}}\right)^{-0.2} = 0.0665^{-0.2}$$

$$(1-\eta|_{TEST}) = (1-\eta|_{FULL})\, 0.0665^{-0.2} = 0.172 \qquad \underline{\underline{\eta_{TEST} = 82.8\%}}$$

$$\Delta P_0|_{FULL} = \rho g(\text{HEAD}) = 1000 \times 9.81 \times 500 = \underline{4.91\ \text{MPa}}$$

$$\text{Power} = \frac{\dot{Q}\,\Delta P_0}{\eta} \qquad \text{Power}|_{FULL} = \frac{50 \times 4.91 \times 10^6}{0.9} = \underline{272.8\ \text{MW}}$$

$$\text{Power}|_{TEST} = \text{Power}|_{FULL} \frac{(\Omega^3 D^5)_{TEST}}{(\Omega^3 D^5)_{FULL}} = 272.8 \left(\frac{3000}{500}\right)^3 \left(\frac{0.4}{3.8}\right)^5 = \underline{\underline{0.761\ \text{MW}}}$$

$$\dot{Q}|_{TEST} = \dot{Q}|_{FULL} \frac{(\Omega D^3)_{TEST}}{(\Omega D^3)_{FULL}} = 50 \left(\frac{3000}{500}\right)\left(\frac{0.4}{3.8}\right)^3 = \underline{0.350\ \text{m}^3/\text{s}}$$

$$\Delta P_0|_{TEST} = \frac{\eta_{TEST}\ \text{Power}|_{TEST}}{\dot{Q}|_{TEST}} = \frac{0.828 \times 0.761 \times 10^6}{0.350} = \underline{1.80\ \text{MPa}}$$

$$\text{HEAD}|_{TEST} = \Delta P_0|_{TEST}/\rho g = 1.80/(1000 \times 9.81) = \underline{\underline{183.5\ \text{m}}} \qquad \boxed{8}$$

Examiner's Comments: Done very badly, candidates could not calculate the test Reynolds number!

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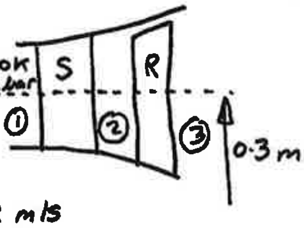
Q8 a) $U = r\Omega = 0.3\left(\frac{2\pi \times 15400}{60}\right) = \underline{483.81\ m/s}$



$$T_2 = \frac{T_{02}}{\left(1+\frac{\gamma-1}{2}M_2^2\right)} = \frac{1200}{\left(1+\frac{0.333}{2}0.9^2\right)} = \underline{1057.4\ K}$$

$$V_2 = M_2\sqrt{\gamma R T_2} = 0.9\sqrt{1.333 \times 287 \times 1057.4} = 572.42\ m/s$$

$$V_{x2} = V_2 \cos\alpha_2 = 572.42 \times \cos 65^\circ = 241.92\ m/s$$

$$V_{\theta 2} = V_2 \sin\alpha_2 = 572.42 \times \sin 65^\circ = 518.79\ m/s$$

FLOW COEFFICIENT $\underline{\underline{\phi = \frac{V_{x2}}{U} = \frac{241.92}{483.81} = 0.500}}$ $\boxed{4}$

b) 50% REACTION ⇒ SYMMETRIC VEL. Δ'S ⇒ $\alpha_3^{REL} = -\alpha_2 = -65^\circ$

$$V_{\theta 3}^{REL} = V_x \tan\alpha_3^{REL} = 241.92 \tan(-65^\circ) = -518.80\ m/s$$

$$V_{\theta 3} = V_{\theta 3}^{REL} + U = -518.80 + 483.81 = -34.99\ m/s$$

$$\psi = \frac{\Delta h_0}{U^2} = \frac{U(V_{\theta 2} - V_{\theta 3})}{U^2} = \frac{518.79 - (-34.99)}{483.81} = \underline{\underline{1.145}}$$ $\boxed{4}$

c) $Y_p = \frac{P_{01} - P_{02}}{P_{02} - P_2} = \frac{P_{01}/P_{02} - 1}{1 - P_2/P_{02}}$ $\qquad \frac{P_{01}}{P_{02}} = 1 + Y_p\left(1 - \frac{P_2}{P_{02}}\right)$

$$\frac{P_2}{P_{02}} = \left(\frac{T_{02}}{T_2}\right)^{-\gamma/\gamma-1} = \left(1+\frac{0.333}{2}0.9^2\right)^{-\frac{1.333}{0.333}} = 0.602640$$

$$\frac{P_{01}}{P_{02}} = 1 + 0.06\left(1 - 0.602640\right) = 1.023842$$

$$P_{02} = \frac{P_{01}}{(\ )} = \frac{8.4}{1.023842} = \underline{\underline{8.204\ bar}}$$ $\boxed{4}$

d) @ $M_2 = 0.9$, $\gamma = 1.333$ TABLES ⇒ $\frac{\dot m\sqrt{c_p T_{02}}}{P_{02} A_2 \cos\alpha_2} = 1.3347$

$$A_2 = \frac{\dot m\sqrt{c_p T_{02}}}{1.3347\, P_{02}\cos\alpha_2} = \frac{40\sqrt{1149 \times 1200}}{1.3347 \times 8.204 \times 10^5 \times \cos 65^\circ} = \underline{0.101499\ m^2}$$

$$A_2 = 2\pi r \times \text{height}$$

$$\text{height} = \frac{0.101499}{2\pi\, 0.3} = 0.0538\,m \qquad = \underline{\underline{53.8\ mm}}$$ $\boxed{4}$

e) STATOR CHOKES WHEN $M_2 = 1$ @ ANGLE 65°

$$\frac{\dot m\sqrt{c_p T_{02}}}{P_{02} A_2 \cos\alpha_2} = 1.3468 \qquad \dot m = \frac{1.3468 \times P_{02} A_2 \cos\alpha_2}{\sqrt{c_p T_{02}}}$$

$$\dot m_{CHOKE} = \frac{1.3468 \times 8.204 \times 10^5 \times 0.101499 \times \cos 65^\circ}{\sqrt{1149 \times 1200}} = \underline{\underline{40.36\ kg/s}}$$ $\boxed{3}$

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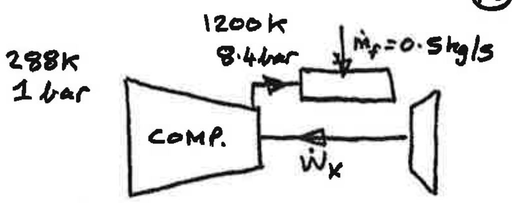
Q8 cont.)

f)



$\dot{m}_c = 40 - 0.5 = 39.5$ kg/s

TURBINE SHAFT POWER

$$= \dot{m}_T C_p \Delta T_0$$

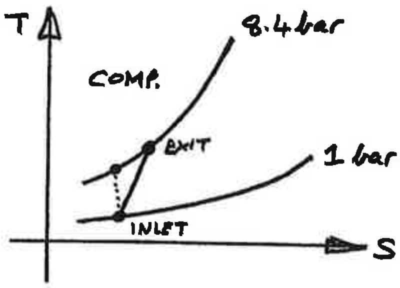
$$\dot{W}_x = 40 \times 1149 \times U^2 \psi$$

$$= 40 \times 1149 \times 483.81^2 \times 1.145$$

$$\underline{\dot{W}_x = 10.721 \text{ MW}}$$

$$\Delta T_0\big|_c = \frac{\dot{W}_x}{\dot{m}_c \, C_p|_c} = \frac{10.721 \times 10^6}{39.5 \times 1005} = \underline{270.1 \text{ K}}$$

$$\frac{T_{0\,EXIT}^{ISEN}}{T_{0\,INLET}} = \left(\frac{P_{0\,EXIT}}{P_{0\,INLET}}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{8.4}{1.0}\right)^{\frac{0.4}{1.4}}$$



$$T_{0\,EXIT}^{ISEN} = 288 \left(\frac{8.4}{1.0}\right)^{\frac{0.4}{1.4}} = 529.02 \text{ K}$$

$$\Delta T_0^{ISEN} = 529.02 - 288 = 241.0 \text{ K}$$

$$\eta_{TT} = \frac{\Delta T_0^{ISEN}}{\Delta T_0} = \frac{241.0}{270.1} = 0.892 \qquad \underline{\underline{89.2\%}}$$

$\boxed{5}$

Examiner's Comments:

a) Done Very well.
b) Candidates had difficulty using symmetric Velocity triangles (50% REACTION) to evaluate the stage loading.
c) Done very well.
d) Common mistake was to omit $\cos\alpha_2$ term.
e) Generally alright but $\cos\alpha_2$ often omitted.
f) Done poorly.

J.P. LONSLEY JUNE 2012.

2012 IIA Paper 3A3 (Fluid Mechanics II) Answers

Q1: a) steady, isentropic, irrotational, linearised.

b) $\left.\frac{\partial \phi}{\partial y}\right|_{y=0} = \frac{\tau}{c} g' u_\infty$

c) $k = -2\frac{\partial \tilde{\phi}}{\partial \tilde{x}}$ where $\tilde{x} = x/c$ and $\tilde{\phi} = \frac{\sqrt{1-M_\infty^2}}{\tau u_\infty}\phi$

d) $u_\infty = 180.7$ m/s $(M_\infty = 0.531)$

e) $u_\infty = 168.3$ m/s $(M_\infty = 0.495)$

Q2: b) $drag / length = 3.82\times10^{-3}\, p_\infty$ N/m

c) 163.8 kW $(drag = 277.5$ N for 6 m long bump)

Q3: b) i) 0.0582 m^2 $(M_{exit} = 2.11)$

ii) 7026 N

c) i) just choked with isentropic subsonic flow in divergent section p_{exit}=1.667 bar

ii) $M_{ahead} = 2.075$ $(p_{0exit} / p_{0inlet} = 0.6856)$

iii) 2397 N

Q4: b) i) $M_{inlet} = 0.3800$, $c_f = 0.0090$ or $M_{inlet} = 1.9792$, $c_f = 0.0010$

c) i) 1 m: $M_{ahead-shock} = 1.7909$, $M_{after-shock} = 0.6186$, $\left.\frac{4c_f L}{D}\right|_{exit} = 0.0322$

2 m: $M_{ahead-shock} = 1.6180$, $M_{after-shock} = 0.6631$, $\left.\frac{4c_f L}{D}\right|_{exit} = -0.0398$

ii) 1.45 m, $M_{ahead-shock} = 1.71$

Q5: c) $h_1/h_0 = 0.444$ (Not a small height change.)

d) $d = 179$ m $(\Delta t = 104.4$ s$)$

Q6: c) Get u_j^{i+1} from momentum equation, then v_{j+1}^{i+1} from continuity

Q7: a) i) Second order in space, first order in time.

ii) $0 < \beta < 0.5$

b) ii) $\left. Power\right|_{test} = 0.761$ MW, $\left.\eta\right|_{test} = 82.8\%$, $\left. Head\right|_{test} = 184$ m $(\left.\dot{Q}\right|_{test} = 0.350$ m^{3}s$^{-1})$

Q8: a) $\phi = 0.500$

b) $\psi = 1.145$

c) 8.20 bar

d) 53.8 mm

e) 40.4 kg/s

f) 89.2%